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A Market Based Solution for Fire Sales and Other Pecuniary Externalities

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Abstract

We show how bundling, exclusivity and additional markets internalize fire sale and other pecuniary externalities. Ex ante competition can achieve a constrained efficient allocation. The solution can be put rather simply: create segregated market exchanges which specify prices in advance and price the right to trade in these markets so that participant types pay, or are compensated, consistent with the market exchange they choose and that type's excess demand contribution to the price in that exchange. We do not need to identify and quantify some policy intervention. With the appropriate ex ante design we can let markets solve the problem.

Keywords: price externalities; segregated exchanges; Walrasian equilibrium; markets for rights to trade; market-based solution; collateral; exogenous incomplete markets; fire sales.

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1 Introduction

Our paper is a tale of several literatures and the importance of bringing them together. One in the wake of the financial crisis is a literature on pecuniary externalities that has regained the interest of researchers as they seek policy interventions and regulations to remedy externality-induced distortions, e.g., balance sheet effects, amplifiers and fire sales. Solutions range from regulation of portfolios, restrictions on saving or credit, interest rate restrictions, fiscal policy, or taxes and subsidies levied by the government[1]. A second literature in the general equilibrium tradition is the exogenous incomplete security markets literature, which shows generically that competitive equilibria are inefficient. However, there is a third literature in general equilibrium theory which dates back to work of one of the founding fathers, Arrow (1969), namely how *bundling*, *exclusivity* and suitably designed *additional markets* can internalize externalities, without the need of further policy interventions (or the need to quantify those interventions). In this world ex ante competition and equilibrium, market-determined prices for rights to trade in these additional markets can achieve a constrained-efficient allocation. The contribution of our paper can be seen as bridging the gap among these literatures and, more importantly, formulating a proposal for an ex ante optimal market design of financial markets which eliminates fire sales inefficiencies and other pecuniary externalities. What we do does have precedents in the literature: Coase (1960) who specifies property rights to remedy externalities (related to the cap and trade idea in pollution) and Lindahl (1958) who uses agent specific prices to solve a public goods problem. We build on these ideas but provide new results.

As pointed out by Lorenzoni (2008) and many others[2], both developed and emerging economies have experienced episodes of rapid credit expansion followed, in some cases, by a financial crisis, with a collapse in asset prices, credit, and investment. As an example, he takes the case of the banking sector in Thailand prior to the crisis of 1997. In the first half of the 1990s, Thai banks increased their investment in real estate. When the crisis erupted,

[1] See, for example, Bianchi (2010); Bianchi and Mendoza (2012); Farhi et al. (2009); Jeanne and Korinek (2013); Korinek (2010).

[2] For the main stylized facts on boom-bust cycles, see Bordo and Jeanne (2002); Gourinchas et al. (2001); Tornell and Westermann (2002), and many others.

the fall in real estate prices eroded the real estate value of those collateral guarantees. This prompted a cut in bank lending, which in turn, led to a further reduction in the demand for real estate and a further drop in real estate prices. This is but one example. There is also a literature on fire sales in New York financial markets, e.g Begalle et al. (2013); Duarte and Eisenbach (2014); Gorton and Metrick (2012); Krishnamurthy et al. (2012).

However, as Lorenzoni (2008) emphasizes, if the private sector had accurate expectations and correctly incorporated risk in its optimal decisions, yet still decided to borrow heavily during booms, it means that the expected gain from increased investment more than compensated for the expected costs of financial distress. Thus one needs to understand how, and under what conditions, this private calculation leads to inefficient decisions at the social level. The answer is that since they are atomistic, private agents do not take into account the general equilibrium effect of asset purchases and subsequent sales on prices. We emphasize here that this general equilibrium effect kicks in only when contracts and markets are limited by obstacles to trade that depend on prices, such as collateral constraints, that the value of a contracted promise to pay cannot exceed the value of the associated collateral which is marked to spot prices. This is the pecuniary externality that is at the basis of the inefficiency result. Indeed in Geanakoplos and Polemarchakis (1986) more generally, the key is that prices enter into constraints beyond budgets so that the set of feasible allocations at the individual level is moving around with the decisions of others as those decisions influence those prices.

Now we go back to Arrow (1969) for some insights about how to remedy such problems. He deals directly with the most obvious specification of a non-pecuniary externality, preferences that depend on what others are consuming. Suppose the utility function of each agent depends not only on her own consumption of each commodity, but also the consumption of commodities of the other agents. Still, in a standard competitive equilibrium, each agent buys her own consumption goods and has no control over the consumption of others, though she does care about this. From the first order condition, taking advantage of the insight that the marginal utility of income in the budget is equal to the inverse Pareto weight in the planning problem, the weighted marginal utility of consumption for a given commodity of each agent is equal to the price of that commodity. In contrast a Pareto optimal allocation of resources can be characterized by maximizing a Pareto weighted sum of utilities subject to resource constraints for each commodity, that consumption not exceed aggregate endowment. As consumption of a given commodity for a given household enters into the utility of all of them, the first- order condition is that the weighted sum of marginal utilities equal the common shadow price of that commodity. Obviously, the first order conditions to these two problems, necessary for competitive equilibrium and optimality, respectively, do not match. The competitive equilibrium is inefficient due to the externality.

Arrow (1969) then writes down an equivalent programming problem for the determination of optima with a subtle shift in notation which, nevertheless, embodies in it a lot of the economics. Namely, Arrow extends the commodity space letting each given agent have the right to specify the consumption of the others, as if buying the consumption of others, so that as far as the given agent is concerned, this looks like a normal private ownership economy without externalities. Notationally, the variables appearing in the utility function relating to the given named agent are proper to him alone and appear in no one else's utility. But this is done for every agent. This then necessitates another equation which specifies that what every single agent wants some other target agent to consume is exactly the same, what all of them want for that agent, and is what that target agent is actually consuming. This consistency in assignments is an explicit extra constraint in the programming problem. Both this and the usual resource constraints pick up Lagrange multipliers. The first is like a price for the right of a given agent to specify consumption of a given commodity for another named agent, and the second is the common price of the underlying commodity. The first order condition for a given agent is that her weighted marginal utility of consumption of the good of the other target agent equals the shadow price in the consistency constraint, the price of rights. The other first order condition is that the sum over all agents of these prices of rights equals the price of the underlying commodity. Thus, by substitution, the sum of weighted marginal utilities for a given commodity of a given agent will equal the common shadow price of that commodity, the same overall optimum condition as before.

The real point is that it is now a short step to decentralize the rewritten optimum problem so that it can be achieved as competitive equilibrium with prices that correspond to the above shadow prices, everyone acting in their own narrow self interest only, not worrying about over all consistency in assignments nor market clearing, though everything is consistent in equilibrium. The problem of an agent as consumer is to maximize utility by specifying consumption not only for herself but for each of the others, and each of these consumptions has a rights price. The sum of expenses cannot exceed wealth. The problem of an agent as supplier is to maximize profits from buying a consumption good and then selling the rights to specify her own consumption to the others, thus receiving the sum of rights prices that each of the others is willing to pay. Essentially, for these concave Lagrangian problems (utilities can be assumed to be concave, resources and assignment constraints are linear, budgets are linear, etc.) first order conditions are not only necessary but also sufficient. Thus the allocations of the decentralized market solution and the centralized planner problem are the same. Both are optimal.

Likewise, as Arrow (1969) does, one might have supposed that each agent cares about the sum of consumption of a given commodity in the population. Then obviously the rights prices for assigning consumption to one target agent is the same as assigning to any other target agent. By extension, one might have supposed that ratios of commodity aggregates enter into utility functions directly. What we do in our first leading example, since we care about pecuniary rather than non-pecuniary externalities, is closely related but a bit different. We let prices enter into collateral constraints and, with homothetic utilities, prices are determined by ratios of commodity aggregates. So in this first example we can naturally refer to the commodity ratio as the market fundamental, as it alone determines prices. Then, to internalize the externality following Arrow, we extend the commodity space so that in the decentralized competitive problem agents can buy the right to specify any ratio of commodity aggregates they want as if they had ownership over that ratio as well as the quantity they wish to buy or sell at that market fundamental. But of course these rights have market prices, as we now explain in more detail.

As a buyer, the object is the number of rights purchased, namely, the fundamental ratio which determines the price and positive excess demand at that price. As a seller, the agent specifies the fundamental ratio and negative excess demand at that associated price, so that such an agent is compensated for agreeing to trade a given market fundamental. The size and sign of the discrepancy, which can be positive or negative, is the difference between an agent's pretrade endowment ratio, inclusive of collateral, and the overall market average[3]. Conceivably an agent's net excess demand at the fundamental could be zero, in which case nothing is paid or received for specifying that fundamental. In general though there will be active trade among some agent types. But the sum of type-weighted discrepancies must by construction be zero (not everyone can be above average in equilibrium). The intuition, again, apart from scale parameters, is that a type's discrepancy is its net excess demand, and consistency thus requires that the type weighted sum of excess demands be zero in any equilibrium. That is, a type's "contribution" to the price is that type's excess demand. Market determined prices will allow agents to pay or be paid for their "contribution".

Importantly, security trades and the rights to trade at spot prices are tied together. A security is a promise made by the issuer to the investor and the associated, specified collateral that goes with the promise. The spot market specifies in addition the value of collateral ex post, the unwind price. That price is now for us here also an intrinsic part of the security.

This is related to but not identical with Lindahl's pricing of public goods. With the latter, each agent can buy the amount of the public good they want at an agent specific price. Under individual agent maximization, that agent's price is equal to that agent's marginal utility gain. A producer of the public good then maximizes revenue as the sum of the per unit prices times the quantity produced less production cost. This yields the sum of marginal utilities equal to the marginal cost, an optimum, achieved in a decentralized market with agent specific prices for rights to specify the quantity of the public good. In equilibrium the amount of the public good is common to all of the agents, but there are agent-specific prices. In contrast, in what we do, the relevant quantity is the agent-specific excess demand, which can be positive, negative, or zero, which can vary over agent types, and in equilibrium must sum to zero. The price per unit excess demand is common, not

[3]This object is related to consumption rights in Bisin and Gottardi (2006), which internalizes the consumption externality due to adverse selection problem. The key difference is that our "discrepancy from the market fundamental" only requires own type information (endowments and savings/collateral position) and the knowledge of the equilibrium price, which is a standard Walrasian assumption, while the determination of the consumption rights for each type in an adverse selection environment utilizes information on other types (see Eq. (3.2) in Bisin and Gottardi (2010) which specifies correct conjectures of what other types are doing and the related no-envy conditions in Prescott and Townsend (1984a,b)).

agent specific, but quantities demanded or supplied are agent specific. Thus agent types have varying levels of expenditures. Bewley (1981) rigorously establishes Tiebout's result but argues that it requires special assumptions and that is not true in general. Lindahl prices can work but in his view they eliminate the essential part of the public goods problem. We note in particular that without Lindahl pricing, without taxes, and with the quantity of public goods in common, expenditure must be the same across agent types.[4] For us, prices are the same across types, so no Lindahl pricing, but quantities, the excess demands, vary because excess demands are different across different types. This makes the point that our formulation is different from public goods or pollution in that these quantities are held in common, whereas in our formulation agent types are buying the rights to a price and to trade their own specific excess demand at that price (which is what influences the equilibrium price).

Equilibrium is achieved by adding a supply side, a set of competitive broker dealers who put together deals with the various agent types and make markets, that is, ensure that in equilibrium supply and demand for securities in the initial market and the demand for rights all net to zero. Indeed, any one broker dealer is of negligible size, and each takes prices as given, so with constant returns to scale in clearing in securities and rights to trade, this generates the equilibrium fees for the market exchanges (platforms) and the rights to trade there. The important point is that the overall Walrasian equilibrium is efficient.

In our second leading example, a classic environment with incomplete securities, spot markets are essential. Dropping homotheticity, we no longer refer to a market fundamental per se; the aggregate excess demand can be a high dimensional and complex mapping from the entire array of individual pre trade endowments. Further, securities need not be state dependent, so a given trade in securities ex ante can have implications for the distribution of income across states; with insufficient ways to hedge, this is precisely why a standard incomplete market equilibrium can be generically inefficient (e.g., Geanakoplos and Polemarchakis, 1986; Greenwald and Stiglitz, 1986). Here, to remove the externality, we do not expand the

[4] This is the reason why the analogy of our approach to pollution, and cap and trade, breaks down. With pollution there is a common level and agents pay for the right to pollute. For us, the problem is two sided, with the position having to do with the influence on the price, and positions must net to zero.

set of securities. We do not complete the markets of securities. Rather we do expand the types of markets. We allow forward contracting in the price vector (with dimension equals to the number of states). A given segregated exchange specifies a given price vector. We allow in principle many possible vectors, hence many possible exchanges. Agent types are buying the rights to trade at these spot prices. We specify the quantities of rights to trade as the vector of excess demands over states given any chosen segregated exchange. These rights are priced so that in equilibrium these exchanges clear. There are active exchanges with trade that clear in the usual sense and inactive exchanges with no trade that clear trivially, with no trade by anyone.

Finally, with these two examples in hand, we generalize our arguments to a large class of environments. These include that of Lorenzoni (2008), which is a cousin to our collateral example; Hart and Zingales (2013), which is a cousin to our incomplete market example; and we extend to information imperfections, a moral hazard contract economy with multiple goods and retrade in spot markets, and a Diamond-Dybvig preference shock economy with retrade in bond markets. In none of these do we actually recover the first best optimum but rather remove the externality and deliver a constrained efficient allocation. The key, and common ingredient across environments, is a set of constraints which contain prices.

The analysis requires excludability (hence the term segregation above). Interestingly, Arrow (1969) is less concerned about excludability, an intrinsic part of creating the necessary markets, as he feels this has a natural counterpart in many, though not all, real world problems. We take this up in our implementation section. Arrow (1969) is more concerned about the obvious small numbers problem. That is, his markets are not thick. There is only one supplier of the own-specific commodity, for example. So while the price-taking assumption of the Walrasian equilibrium allows one to carry out the analysis above, it is not realistic. However, this part is easy to remedy if we consider limit economies with a continuum of traders and positive mass of each traders type, identical in preferences and endowments within type.[5]

[5]In both the optimum problem and the decentralized problem we restrict attention to type symmetric allocations (a core allocation has to have this property or otherwise it can be blocked and the maximization problem facing each agent of a given type is identical. Then the populations weights are in both sides of

There is, however, an assignment problem that allows yet another extension of the commodity space and reinforces the notion of excludability. Due to some inherent nonconvexities in the way prices enter into constraints, namely collateral times the price of collateral as in our first example economy, it may be necessary to convexify the problem by allowing mixtures. These are priced so that in the decentralized problem each agent can control the probability that they are assigned to given markets and are excluded from others. What is a probability in the way an individual of a given type is treated from the individual point of view is also a fraction of the way that type is treated in the aggregate. The continuum sets of agent types allows us to do this. Thus resource constraints, market clearing, and broker dealer technologies are relatively easy to write down and do not involve lotteries. The lotteries at the individual level, and mixtures at the aggregate level, are part of our technical apparatus, working for us much the way it does for Koopmans and Beckmann (1957); Prescott and Townsend (1984a,b); Prescott and Townsend (2006).

We emphasize also that our proofs in space of mixtures are standard in that preferences, upper contour sets, are convex and constraints are linear. Further, in many applications these individual lotteries are degenerate, that is, agents purchase deterministic assignments, so the notation of a lottery is essentially only an indicator function. But in other environments they do play an active role, and we display an example.

The remainder of the paper proceeds as follows. Section 2 presents two key leading examples including a collateral economy and an exogenous incomplete markets economy. Section 3 describes briefly how to map a large variety of example economies into a generalized framework in which our market-based solution concept is applicable. We discuss how to implement our market-based solution in the financial markets in Section 4. Section 5 concludes. Appendix A generalizes to mixtures and Appendix B contains some formal proofs. Additional results are contained in Appendix C.

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first order conditions and cancel out.) The intuition above remains exactly the same.

2 Leading Examples

This section features two example economies, a collateral economy (Kilenthong and Townsend, 2014b) and an exogenous incomplete markets economy (Geanakoplos and Polemarchakis, 1986; Greenwald and Stiglitz, 1986).

2.1 A Collateral Constrained Economy

This is a two-period economy, $t = 0, 1$. All financial (debt and insurance) contracts are traded in period $t = 0$, henceforth called the "contracting period". In addition, in period $t = 0$, both of two consumption goods can be traded and consumed. All contracts will be executed in period $t = 1$, henceforth called the "execution period". The two goods are also traded there. There are a finite number S of possible states of nature in this execution period $t = 1$, i.e., $s = 1, 2, ..., S$. This allows as a special case $S = 1$, so there is only intertemporal trade, borrowing and lending, from $t = 0$ to $t = 1$. For $S > 1$ in which contingent claims, Arrow-Debreu securities are traded, let $0 < \pi_s < 1$ be the objective, commonly assessed, and actual probability of state s occurring, where $\sum_s \pi_s = 1$. Again, the two underlying goods can be traded and consumed in each state s. We refer to these $t = 1$ markets as spot markets.

The two underlying goods are called good 1 and good 2. Good 1 cannot be stored (is completely perishable), while good 2 is storable from $t = 0$ to $t = 1$. Good 2 can serve as collateral to back promises issued in the contracting period. Henceforth, good 2 and the collateral good will be referred to interchangeably. Furthermore, good 1 will be the numeraire good in every date and state. The price in terms of the numeraire at which the collateral good 2 can be unwound in spot markets is the key object associated with the pecuniary externality.

There is a continuum of agents of measure one. The agents are however divided into H heterogeneous types, each of which is indexed by $h = 1, 2, \cdots, H$. Each type h consists of $\alpha^h \in (0, 1)$ fraction of the population, so that $\sum_h \alpha^h = 1$. Each agent type h is endowed with good 1 and good 2, $\mathbf{e}_0^h = \left(e_{10}^h, e_{20}^h\right) \in \mathbb{R}_+^2$ in period $t = 0$ and $\mathbf{e}_s^h = \left(e_{1s}^h, e_{2s}^h\right) \in \mathbb{R}_+^2$, in each state $s = 1, \cdots, S$ at period $t = 1$. Let $\mathbf{e}^h = \left(\mathbf{e}_0^h, \cdots, \mathbf{e}_S^h\right) \in \mathbb{R}_+^{2(1+S)}$ be the entire endowment profile of agent type h over period $t = 0$ and all states s in period $t = 1$. As a notational convention, vectors or matrices will be represented by bold letters. Heterogeneity of agents originates in part from the endowment profiles $\mathbf{e}^h$ (and not in preferences, but we could easily allow this extension). We also assume that the endowments in all periods and all states are publicly known. Hence, the limited commitment considered in this paper comes from a contract enforceability problem, not from an informational problem. But see our comments in Section 3 on generalizations.

Let $k^h \in \mathbb{R}_+$ denote the collateral holding (equivalent to the holding of good 2) of an agent type h at the end of period $t = 0$ to be carried to period $t = 1$. Note that this collateral allocation does not need to be equal to the initial endowment of good 2, e_{20}^h. In particular, since good 2 can be exchanged or acquired in the contracting period (at date $t = 0$), k^h will be equal to the net-position in the collateral good after trading in period $t = 0$. The collateral good as legal collateral backing claims is assumed to be fully registered and kept in escrow, i.e., cannot be taken away or stolen neither by borrowers nor lenders. However, the holding of good 2 can also include normal saving. The storage technology of good 2, whether in collateral or normal savings, is linear but with a potentially random return. In some applications, it is natural to treat the returns as a constant and focus on how collateral interacts with intertemporal trade. In other applications, the risk is in the collateral itself. Each unit of good 2 stored will become R_s units of good 2 in state $s = 1, \cdots, S$. Specifically, storing I units of good 2 at date $t = 0$ will deliver $R_s I$ units of good 2 in state s at $t = 1$.

The contemporary preferences of agent type h are represented by the utility function $u\left(c_1^h, c_2^h\right) : \mathbb{R}_+^2 \to \mathbb{R}$, which is assumed to be continuous, strictly concave, strictly increasing in both consumption goods, and to satisfy the usual Inada conditions. Let $0 < \beta \leq 1$ be the common discount factor. The discounted expected utility of agent type h is thus

$$U^h\left(\mathbf{c}^h\right) \equiv u\left(c_{10}^h, c_{20}^h\right) + \beta \sum_{s=1}^{S} \pi_s u\left(c_{1s}^h, c_{2s}^h\right)$$

where, as with the notation for endowments, $\mathbf{c}^h = \left(\mathbf{c}_0^h, \cdots, \mathbf{c}_S^h\right) \in \mathbb{R}_+^{2(1+S)}$ is the consumption allocation.

For full generality here, we will consider state-contingent securities as the primitives and otherwise let the security structure be endogenous. That is, we are dealing with an Arrow-

Debreu complete security environment, but collateral will limit the securities which emerge in equilibrium.[6]

We consider at most only two classes of securities[7]; (i) θ^h_{1s} - securities paying in good 1 in state s, (ii) θ^h_{2s} - securities paying in good 2 in state s. Here a positive number denotes the purchaser or holder, and negative the issuer, the one making the promise. When negative, each of the state-contingent securities must be backed. The collateral constraints for an agent type h thus take the intuitive form

$$p_s R_s k^h + \theta^h_{1s} + p_s \theta^h_{2s} \quad \geq \quad 0, \ \forall s. \tag{1}$$

The collateral constraint (1) states that, for each state s, the net-value of all assets, including collateral good and securities, must be non-negative. If θ^h_{1s} and θ^h_{2s} were negative, as promises, we could write this as $p_s R_s k^h \geq -\theta^h_{1s} - p_s \theta^h_{2s}$. That is, there is sufficient collateral in value in state s to honor the value of all such promises. Note that all promises are converted to units of good 1 using the spot market price of the collateral good p_s.

2.1.1 Competitive Collateral Equilibrium (with an externality)

Agents can trade in spot markets, and let $\tau^h_{\ell s}$ be the amount of good $\ell = 1, 2$ bought by an agent type h in the spot markets at state s at prices p_s, respectively. Let p_0 be the price

[6] A specific piece of collateral can be used to back up several contracts as long as their promises to pay are in different states. Thus there is no conflict in a given state s. This is known as *tranching.* This is distinct from the contract-specific collateralization structure (in Geanakoplos, 2003, among others), in which the collateral of a given security cannot be used as collateral for any other security. A security which would default has a known payoff structure, so we may as well start with that in the first place. So it can be shown that there is no loss of generality in restricting attention to securities without default. But the possibility of default does restrict securities, and collateral constraints can be binding. Further, issuing securities that do default requires no less collateral than (an equivalent set of) securities that do not. See Kilenthong and Townsend (2014b) for more details.

[7] Actually with spot markets we need securities θ^h_{1s} paying in the numeraire only. We proceed here in more generality as what we do will not require active spot markets. As shown in Kilenthong and Townsend (2014b), spot markets are redundant when all types of state contingent contracts are available ex-ante. In other words, agents do not really need to trade in spot markets even though they may well do so. But promises in ex ante markets still need to be backed by collateral in escrow.

of good 2 in period $t = 0$, and $Q_{\ell s}$ be the prices of securities in the ex ante $t = 0$ market paying in good $\ell = 1, 2$ in state s, respectively, all priced in the numeraire good 1 at $t = 0$. For notational convenience, let $\mathbf{p} \equiv (p_1, \ldots, p_S)$ and $\mathbf{Q} \equiv [Q_{\ell s}]_{\ell,s}$. A collateral equilibrium is thus defined:

Definition 1. A competitive collateral equilibrium is a specification of prices $(p_0, \mathbf{Q}, \mathbf{p})$, and an allocation $\left(\mathbf{c}_0^h, k^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)_h$ such that

(i) for any agent type h as a price taker, $\left(\mathbf{c}_0^h, k^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)$ solves

$$\max_{\left(\mathbf{c}_0^h, k^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)} u\left(c_{10}^h, c_{20}^h\right) + \beta \sum_s \pi_s u\left(e_{1s}^h + \theta_{1s}^h + \tau_{1s}^h, e_{2s}^h + R_s k^h + \theta_{2s}^h + \tau_{2s}^h\right) \tag{2}$$

subject to the collateral constraints (1) for each state s, and the budget constraint at $t = 0$:

$$c_{10}^h + p_0\left(c_{20}^h + k^h\right) + \sum_s Q_{1s}\theta_{1s}^h + \sum_s Q_{2s}\theta_{2s}^h \leq e_{10}^h + p_0 e_{20}^h, \tag{3}$$

and the spot budget constraint at each state s:

$$\tau_{1s}^h + p_s \tau_{2s}^h = 0, \tag{4}$$

(ii) markets clear for good 1 at $t = 0$, for good 2 at $t = 0$, for $\theta_{\ell s}^h$ in state s, and for $\tau_{\ell s}^h$ in state s, respectively:

$$\sum_h \alpha^h c_{10}^h \leq \sum_h \alpha^h e_{10}^h, \tag{5}$$

$$\sum_h \alpha^h \left[c_{20}^h + k^h\right] \leq \sum_h \alpha^h e_{20}^h, \tag{6}$$

$$\sum_h \alpha^h \theta_{\ell s}^h = 0, \ \forall s; \ell = 1, 2; \tag{7}$$

$$\sum_h \alpha^h \tau_{\ell s}^h = 0, \ \forall s; \ell = 1, 2; \tag{8}$$

The necessary maximizing condition for a collateral equilibrium (ce) related to collateral allocation k^h (an interior solution to the consumer problem) is given by, for any h,

$$p_0 = \left.\frac{u_{20}^h}{u_{10}^h}\right|_{ce} = \sum_s \pi_s \beta \frac{u_{2s}^h}{u_{10}^h} R_s + \sum_s \frac{\gamma_{cc-s}^h}{u_{10}^h} p_s R_s, \tag{9}$$

where $u_{\ell 0}^h = \frac{\partial u\left(c_{10}^h, c_{20}^h\right)}{\partial c_{\ell 0}}$, $u_{\ell s}^h = \frac{\partial u\left(c_{1s}^h, c_{2s}^h\right)}{\partial c_{\ell s}}$ for $\ell = 1, 2$, and γ_{cc-s}^h is the Lagrange multiplier for the collateral constraint (1) in state s for an agent type h.

2.1.2 Homothetic Preferences

In principle, the market-clearing prices in these spot markets depend on the distribution of pretrade (before ex post spot trade) endowments across agents. So to generate intuition and to allow closed-form solutions we make a strong simplifying assumption, namely homotheticity. With identical homothetic preferences, the aggregate ratio of good 1 to good 2 in state s is the market fundamental in state s, z_s, that determines price $p_s = p(z_s)$; that is,

$$z_s = \frac{\sum_h \alpha^h e_{1s}^h}{R_s K + \sum_h \alpha^h e_{2s}^h}, \tag{10}$$

where $K = \sum_h \alpha^h k^h$ is the aggregate (endogenous) saving including collateral. This is where we exploit the homotheticity assumption; ratios of the aggregate are enough to pin down equilibrium prices p_s.

Now let $\Delta_s^h = z_s \left(e_{2s}^h + R_s k^h\right) - e_{1s}^h$ denote agent type h's contribution to the equilibrium price. The discrepancy $\Delta_s^h(z_s)$ thus reflects the good-2-weighted gap between the market pretrade ratio and that of type h, namely, rewriting,

$$\Delta_s^h(z_s) = \left(e_{2s}^h + R_s k^h\right)\left(z_s - \frac{e_{1s}^h}{e_{2s}^h + R_s k^h}\right). \tag{11}$$

This expression can be positive if k^h is large so that agent type h is adding a lot of the collateral good in state s, in effect lowering the price, or negative if say type h is well endowed with good 1, raising the price. Finally note that $\Delta_s^h(z_s)$ for type h is a function of both the market fundamental z_s and type h's own pretrade endowment, which if different from z_s will determine the extent of spot trade. If the discrepancy $\Delta_s^h(z_s)$ is positive there is net excess demand for good 1 and net supply of good 2. In addition, summing over h, weighted by mass of type h, α^h, yields the following market clearing condition:

$$\sum_h \alpha^h \Delta_s^h = z_s \sum_h \alpha^h \left(e_{2s}^h + R_s k^h\right) - \sum_h \alpha^h e_{1s}^h = 0, \tag{12}$$

where the last equality results from condition (10). In the ultimate decentralization below, this will be a clearing condition for the rights to trade. It is the inclusion of both the fundamental and the extent and direction of trade under that fundamental that removes externalities. Both will be chosen by each agent type. This relates back to Arrow (1969).

2.1.3 Collateral Constrained Optimality

Attainable allocations are those that can be achieved by exchanges of securities and collateral in date $t = 0$ as well as exchanges of consumption goods in date $t = 1$ at state s, but respecting that agents can trade good 1 for good 2 freely in each state s, and the planner cannot prevent it[8]. In other words, a planner can only reallocate goods with the same instruments as the agents, the security holdings and promises, and must respect the possibility of active spot market trade. Thus the planner must respect the associated collateral constraints assigning collateral to promises but allowing collateral to be unwound at the implicit or explicit spot price inherent in the proposed allocation[9]. Accordingly, attainable allocations are defined using the spot-price function $p(z_s)$. Likewise Geanakoplos and Polemarchakis (1986) search for a Pareto improving allocation of assets recognizing the influence of an asset allocation on spot prices, that spot prices are determined by the conditions that aggregate excess demands must equal zero.[10]

Definition 2. An allocation $\left(\mathbf{c}_0^h, k^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)_h$ is *attainable* if

(i) it satisfies resource constraints (5)-(8);

(ii) for each agent type h, it satisfies the collateral constraints for each state s:

$$p(z_s)R_s k^h + \theta_{1s}^h + p(z_s)\theta_{2s}^h \quad \geq \quad 0, \ \forall s, \tag{13}$$

and the type h spot budget constraints

$$\tau_{1s}^h + p\left(z_s\right)\tau_{2s}^h = 0, \forall s; \tag{14}$$

(iii) the consistency constraint (10) holds for all s.

A constrained optimal allocation is characterized using the following planner's problem. Let $\overline{U}^h$ be the reservation utility level for an agent type h.

[8] An individual agent, in contrast, has zero mass and no influence on prices regardless of the market structure.

[9] Or using a forward price ratio to value promises made in good 1 in terms of good 2, as in footnote 7.

[10] See Section 7.2, page 87 in that paper.

Program 1. The Pareto Program with collateral constraints:

$$\max_{\left(\left(\mathbf{c}_0^h, k^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)_h, z_s\right)} u\big(c_{10}^1, c_{20}^1\big) + \beta \sum_s \pi_s u\big(e_{1s}^1 + \theta_{1s}^1 + \tau_{1s}^1, e_{2s}^1 + R_s k^1 + \theta_{2s}^1 + \tau_{2s}^1\big) \tag{15}$$

subject to (5)-(8), (10), (13)-(14) and the participation constraint for each $h = 2, \cdots, H$,

$$u\big(c_{10}^h, c_{20}^h\big) + \beta \sum_s \pi_s u\big(e_{1s}^h + \theta_{1s}^h + \tau_{1s}^h, e_{2s}^h + R_s k^h + \theta_{2s}^h + \tau_{2s}^h\big) \;\; \geq \;\; \overline{U}^h, \tag{16}$$

and non-negativity constraints for consumption and collateral allocations.

As is typically the case, it suffices to consider only equal-treatment-of-equals in the Pareto problem[11]. Let μ_{cc-s}^h, and $\mu_{\bar{u}}^h$ denote the Lagrange multipliers for the collateral constraint (13) for agent h in state s, and for the participation constraint (16) for agent h, respectively. For notational convenience, let $\mu_{\bar{u}}^1 = 1$. A necessary condition[12] for constrained optimality (op) related to collateral allocation k^h is given by, for any h,

$$\left.\frac{u_{20}^h}{u_{10}^h}\right|_{op} = \sum_s \pi_s \beta \frac{u_{2s}^h}{u_{10}^h} R_s \;\; + \;\; \sum_s \frac{\mu_{cc-s}^h}{\mu_{\bar{u}}^h u_{10}^h} p(z_s) R_s - \sum_s \frac{\alpha^h}{\mu_{\bar{u}}^h u_{10}^h} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \theta_{1s}^{\tilde{h}}, \tag{17}$$

where $p'(z_s) = \frac{\partial p(z_s)}{\partial z_s}$, $K = \sum_h \alpha^h k^h$.

2.1.4 The Externality

Note that an infinitesimal agent of type h takes a spot price, $p(z_s)$, as invariant to his or her own actions in the collateral equilibrium. To the contrary, the constrained planner

[11] Again, for exposition simplicity and without any real loss, we consider only equal-treatment (for each type), and interior solutions (i.e., the non-negativity constraint for k^h is neglected). With homothetic and strictly concave preferences, and no non-convexity, agents of the same type will optimally choose the same allocation in an equilibrium; that is, given the same market prices in equilibrium. Thus, a collateral equilibrium allocation has equal treatment of equals property. More generally, externalities in this class of models, if they exist, have nothing to do with the equal treatment of equals property.

[12] Given that the constraint set is not convex, this optimality condition is necessary but may not be sufficient. Nevertheless, this does not cause any problem to our externality argument, as we simply need to show that a collateral equilibrium cannot be constrained optimal, i.e. does not satisfy the necessary optimal condition (17).

can influence the spot prices $p(z_s)$ through collateral assignments for the agents of type $h = 1, 2, \ldots, H$, in period $t = 0$, namely k^h, which affect in turn the market fundamentals z_s. This key influence is the term in $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K}$. If the very last term in (17) were zero and we set $\gamma^h_{cc-s} = \frac{\mu^h_{cc-s}}{\mu^h_{\bar{u}}}$, then condition (9) is exactly the same as (17), and there would be no externality. The last term in (17) could be zero if either $\mu^{\tilde{h}}_{cc-s} = 0$, that is, no collateral constraint is binding for any $\tilde{h}$ or $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} = 0$.[13]

Proposition 1. *With continuous, strictly concave, strictly increasing, and identical homothetic utility functions, a competitive collateral equilibrium is constrained optimal if and only if all collateral constraints are not binding, i.e.* $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ *for all* h *and all* s.

Proof. See Appendix C. □

In particular when the very last term in (17) is not zero so not first-best, we can show that it must be positive. As a result, the equilibrium price of good 2 in period $t = 0$ will be too high relative to its shadow price from the (constrained) optimal allocation $\left.\frac{u^h_{20}}{u^h_{10}}\right|_{op}$. In addition, this implies that the competitive collateral equilibrium level of (endogenous) aggregate saving K^{ce} is too large[14] relative to the (constrained) optimal level of aggregate saving/collateral K^{op}. Intuitively, the planner can do better by lowering the aggregate saving/collateral. The result is summarized in the following proposition.

Proposition 2. *With continuous, strictly concave, strictly increasing, and identical homothetic utility functions, if a competitive collateral equilibrium is not first-best optimal, then*

(i) the equilibrium price of good 2 in period $t = 0$, p_0, *is too high, i.e.,* $p_0 > \left.\frac{u^h_{20}}{u^h_{10}}\right|_{op}$, *and*

[13] But under homothetic preferences the later is not possible. With a strictly concave utility function, the spot price varies with the market fundamental (is not constant), i.e. $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \neq 0$. As a result, when at least one of the collateral constraints is binding, i.e., $\mu^{\tilde{h}}_{cc-s} > 0$ for some type $\tilde{h}$, the last term in (17) will be non-zero. With this non-zero term, a collateral equilibrium will not be constrained efficient. It is true that, as an exceptional case, a collateral equilibrium could be a full first-best optimum, that is, the environment could be such that despite the focus of the paper we could ignore the collateral constraint.

[14] This is our analogous here to the result of Hart and Zingales (2013) that it is possible for agents to be saving too much.

(ii) the (endogenous) aggregate saving/collateral in a competitive collateral equilibrium, K^{ce}, is too large, i.e., $K^{ce} > K^{op}$.

Proof. See Appendix C. □

This is also our most simple version of Lorenzoni (2008) and the fire sales model. There is too much debt and hence too much collateral, and this is moving (distorting) prices.

2.1.5 New Markets for the Rights to Trade in Segregated Exchanges and Their Prices

The externality problem is in general a missing-market problem. For us, here in this example, the missing markets are those over the "market fundamentals"; that is, those aspects of the environment which determine the spot-market-clearing price. The forces determining the valuation of collateral, are as yet, in the previous section, not contracted.[15] Therefore, our solution is to create new markets for rights to trade. The right for type h in state s at market fundamental z_s is the discrepancy from the fundamental[16], as discussed earlier, namely, $\Delta_s^h(z_s) = \left(e_{2s}^h + R_s k^h\right)\left(z_s - \frac{e_{1s}^h}{e_{2s}^h + R_s k^h}\right)$. In effect, agent types choose the fundamental z_s, or price, at which they want to trade and also the amount they will trade at that z_s. This discrepancy when priced will make each type pay or be paid according to the marginal impact

[15] It is true when there are states of the world, agents are implicitly trading securities that are contingent on realized prices. But what we do is different. Suppose, as in our Environment 1 in Section 2.1.6, there is no uncertainty, hence nothing contingent. The future spot price is a known, deterministic number. There remains an externality problem, however, through the collateral constraints. Hence we allow agents to commit to trade at different alternative spot prices, essentially any price they want and can afford. The prices of these markets then guide agents to a new equilibrium which is constrained efficient, with a deterministic price which is different from the original one.

[16] In the general model of Kilenthong and Townsend (2014a), much like the consumption constraints in Arrow (1969), rights to trade can be determined by excess demand functions. In fact, we can map Δ_s^h into the excess demand as follows. For example, with the following common CRRA utility function $u(c_1, c_2) = \frac{c_1^{1-\gamma}-1}{1-\gamma} + \frac{c_2^{1-\gamma}-1}{1-\gamma}$, we can show that the excess demand function for good 1 with the market fundamental z_s is given by $\tau_1^{h*}(z_s) = \left(\frac{z_s^{\gamma-1}}{1+z_s^{\gamma-1}}\right)\Delta_s^h(z_s)$. Note that the additional term $\frac{z_s^{\gamma-1}}{1+z_s^{\gamma-1}}$ depends on z_s only so, to go back and forth between discrepancy and excess demand, we have to only redefine units traded in each state s. A key advantage of the discrepancy Δ_s^h defined in (11) is the independence from the utility function form.

of that type on the price. These prices of rights will come from the planner problem and reflect the marginal valuation of the market-clearing constraint (12), $\sum_h \alpha^h \Delta_s^h(z_s) = 0$.

Let $P_\Delta(z_s, s)$ be the unit price of rights to trade at spot-market price $p_s = p(z_s)$, where the latter is the price used to unwind collateral. The fee for the rights to trade in a security exchange z_s is given by that price $P_\Delta(z_s, s)$ times the quantity of the discrepancy, $\Delta_s^h(z_s)$, namely the total expenditures is $P_\Delta(z_s, s)\,\Delta_s^h(z_s)$ (or revenue if negative).

To internalize the externality, agents who bring in too much of good 2 relative to the market fundamental would have to pay for rights to trade, and vice versa. For example, if $\Delta_s^h(z_s) > 0$, then $z_s > \frac{e_{1s}^h}{e_{2s}^h + R_s k^h}$ and type h holds a relative low amount of good 1 and abundant amount of good 2 relative to z_s. As a result, that type h would need to pay for the right to trade or unwind in this market. This makes intuitive sense since the problem, the inefficiency, is oversaving. Conversely, when $\Delta_s^h(z_s) < 0$, an agent type h has a relatively high amount of good 1 and scarce amount of good 2, relative to z_s. With oversaving in the aggregate this type will be compensated.

Note that if type h's pretrade endowment is exactly equal to the market fundamental, $\frac{e_{1s}^h}{e_{2s}^h + R_s k^h} = z_s$, then $\Delta_s^h(z_s) = 0$. But typically with heterogeneity and active trade an agent type h will be on one side or the other of the market fundamental, buying or selling the collateral good 2 for good 1. Of course it takes at least two sides to open an active market. Finally, an agent type h is making these commitments over all states s, so the total impact on its budget in the contracting period is $\sum_s P_\Delta(z_s, s)\,\Delta_s^h(z_s)$.

Competitive Equilibrium with Segregated Exchanges

Each agent must choose one but only one fundamental spot market in each state s, that is, one forward price at which collateral will be unwound. More formally, let an indicator function $\delta^h(z_s) \in \{0, 1\}$, that is, $\delta^h(z_s) = 0$ or $\delta^h(z_s) = 1$, denote an agent type h's discrete choice of spot and security market z_s in each state $s = 1, 2, \ldots, S$. With vector $\mathbf{z} = (z_s)_{s=1}^S$, we write this function as $\delta^h(\mathbf{z})$ so that the particular z_s in each s is specified. These choices of $\mathbf{z}$ are jointly bundled with spot trade $\tau_{\ell s}^h$ in state s, rights $\Delta^h(z_s)$ and securities $\theta_{\ell s}^h$.[17] In

[17]It may seem the consumption goods are also indexed but in fact they are the residual implied by the choice of securities $\theta_{\ell s}^h(\mathbf{z})$ and the associated collateral $k^h(\mathbf{z})$ at $t = 0$. However, spot markets for consumption and

sum, notationally, let $x^h(\mathbf{z}) = \left(\delta^h(\mathbf{z}), \mathbf{c}_0^h(\mathbf{z}), k^h(\mathbf{z}), \boldsymbol{\theta}^h(\mathbf{z}), \boldsymbol{\tau}^h(\mathbf{z}), \boldsymbol{\Delta}^h(z_s)\right)$ denote a typical bundle or allocation for an agent type h, where $\boldsymbol{\Delta}^h(\mathbf{z}) \equiv \left[\Delta_s^h(z_s)\right]_s$. Again the entire vector is now indexed by choice of $\mathbf{z}$ and the latter is captured by $\delta^h(\mathbf{z})$.[18] Likewise prices other than p_0 will be indexed by z_s. Let $Q_\ell(z_s, s)$ denote the price of security $\theta_{\ell s}^h(\mathbf{z})$ with with $\mathbf{Q} \equiv [Q(z_s, s)]_{s, z_s}$, and $P_\Delta(z_s, s)$ denote the market price of rights to trade in security exchange z_s in state s, the price of rights $\Delta_s^h(z_s)$ with $\mathbf{P}_\Delta \equiv [P_\Delta(z_s, s)]_{s, z_s}$.

Otherwise, apart from the bundling and these prices, a competitive equilibrium with segregated exchanges is similar to Definition 1 except that the objective function and constraints are premultiplied by the discrete choice, the budget is augmented by expenditures on rights and there is an additional clearing equation for these rights.

Definition 3. A competitive equilibrium with segregated exchanges is a specification of allocation $\left[x^h(\mathbf{z})\right]_{h,\mathbf{z}}$ and prices $(p_0, \mathbf{Q}, \mathbf{P}_\Delta, \mathbf{p})$ such that

(i) for any agent type h as a price taker, allocation $\left[x^h(\mathbf{z})\right]_{\mathbf{z}}$ solves

$$\max_{\left[x^h(\mathbf{z})\right]_{\mathbf{z}}} \sum_{\mathbf{z}} \delta^h(\mathbf{z}) \Big[u\left(c_{10}^h(\mathbf{z}), c_{20}^h(\mathbf{z})\right) + \sum_s \pi_s u\left(e_{1s}^h + \theta_{1s}^h(\mathbf{z}) + \tau_{1s}^h(\mathbf{z}), e_{2s}^h + R_s k^h(\mathbf{z}) + \theta_{2s}^h(\mathbf{z}) + \tau_{2s}^h(\mathbf{z})\right)\Big]$$

subject to collateral constraints

$$\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \left[p(z_s)\left[R_s k^h(\mathbf{z}) + \theta_{2s}^h(\mathbf{z})\right] + \theta_{1s}^h(\mathbf{z})\right] \geq 0, \forall s, \tag{18}$$

budget constraint at $t = 0$

$$\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \Big\{c_{10}^h(\mathbf{z}) + p_0\left[c_{20}^h(\mathbf{z}) + k^h(\mathbf{z})\right] + \sum_s \sum_\ell Q_\ell(z_s, s)\,\theta_{\ell s}^h(\mathbf{z}) + \sum_s P_\Delta(z_s, s)\,\Delta_s^h(z_s)\Big\} \leq e_{10}^h + p_0 e_{20}^h, \tag{19}$$

and budget constraint at state s

$$\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \left[\tau_{1s}^h(\mathbf{z}) + p(z_s)\,\tau_{2s}^h(\mathbf{z})\right] = 0, \forall s; \tag{20}$$

collateral at $t = 0$ are not segregated in that everyone can trade consumptions including the collateral good together in the $t = 0$ market.

[18] If $\delta^h(\mathbf{z}) = 0$, then the rest need not be specified.

(ii) markets clear for good 1 in period $t = 0$, for good 2 in period $t = 0$, for securities, spot trades, and for rights to trade, respectively,

$$\sum_h \sum_{\mathbf{z}} \delta^h(\mathbf{z}) \alpha^h c_{10}^h(\mathbf{z}) = \sum_h \alpha^h e_{10}^h, \tag{21}$$

$$\sum_h \sum_{\mathbf{z}} \delta^h(\mathbf{z}) \alpha^h \left[c_{20}^h(\mathbf{z}) + k^h(\mathbf{z})\right] = \sum_h \alpha^h e_{20}^h, \tag{22}$$

$$\sum_h \sum_{\mathbf{z}_{-s}} \delta^h(\mathbf{z}) \alpha^h \theta_{\ell s}^h(\mathbf{z}) = 0, \forall s, \ell, z_s, \tag{23}$$

$$\sum_h \sum_{\mathbf{z}_{-s}} \delta^h(\mathbf{z}) \alpha^h \tau_{\ell s}^h(\mathbf{z}) = 0, \forall s, \ell, z_s, \tag{24}$$

$$\sum_h \sum_{\mathbf{z}_{-s}} \delta^h(\mathbf{z}) \alpha^h \Delta_s^h(z_s) = 0, \forall s, z_s, \tag{25}$$

where $\mathbf{z}_{-s} = (z_1, \ldots, z_{s-1}, z_{s+1}, \ldots, z_S)$ is a vector of market fundamentals in all states but state s.[19]

Note that from the market clearing condition of the rights to trade (25), for any active spot market chosen by multiple types, spot markets must clear and in this sense the valuation of collateral is self-fulfilling.

Public Finance Interpretation

The budget constraint with the prices of the rights to trade has a public finance interpretation, as if we were to try to implement the optimum solution by taxes and subsidies.

Specifically, substituting $\Delta_s^h(z_s) = z_s\left(e_{2s}^h + R_s k^h\right) - e_{1s}^h$ into the budget constraint for an agent type h, equation (19) gives

$$\begin{aligned}\sum_{\mathbf{z}} \delta^h(\mathbf{z}) \Big\{ c_{10}^h(\mathbf{z}) + p_0\left[c_{20}^h(\mathbf{z}) + k^h(\mathbf{z})\right] + \sum_s \sum_\ell Q_\ell(z_s, s)\, \theta_{\ell s}^h(\mathbf{z}) \leq e_{10}^h + p_0 e_{20}^h \\ - \sum_s \left[P_\Delta(z_s, s) z_s\right] e_{2s}^h - \sum_s \left[-P_\Delta(z_s, s)\right] e_{1s}^h - \Big[\sum_s P_\Delta(z_s, s) z_s R_s\Big] k^h \Big\}\end{aligned}$$

We can now see that we need to have three types of taxes/subsidies, (i) saving/collateral tax of $\sum_s P_\Delta(z_s, s) z_s R_s$ per unit of saving/collateral, (ii) state-contingent collateral good

[19] Note that (21)-(22) are summed over all $\mathbf{z}$ while (23)-(25) are conditional on partition z_s. The difference reflects the fact that markets for good 1 and good 2 in period $t = 0$ are not segregated but the others are.

endowment tax of $P_{\Delta}(z_s, s)\, z_s$ per unit of collateral good endowment in state s, and (iii) state-contingent subsidy, negative tax $-P_{\Delta}(z_s, s)$ per unit of consumption good endowment of good 1 in state s. Note that it is not sufficient to tax/subsidize only saving/collateral. The tax/subsidy rate on endowments also depends on the security exchange z_s. That is, the security exchange z_s itself is a choice as far as the household is concerned. This is like looking up marginal rates in a big tax book and settling on which page (or pages) to use, indexed by the exchanges z_s that the agent chooses.

But again we do not need the taxes. We let the markets decide. Markets determine prices and prices determine allocations.

2.1.6 Example Economies with Segregated Security Exchanges

Before we generalize, and prove the welfare theorems, we present a series of three examples. The first illustrates the basic mechanics of rights to trade we have featured so far in the simplest possible setting without uncertainty (in which in equilibrium there are no securities beyond simple saving). The second features two states of the world in which insurance contracts are actively traded in both the equilibrium with and without the externalities. The third is an example which has mixtures at the aggregate level and lotteries at the individual level, and this provides an illustrative transition to the more general notation in Appendix A, and proofs of the welfare theorems in Appendix B.

Environment 1 (Intertemporal Smoothing)**.** There are two periods, $t = 0, 1$, and a single state, $S = 1$ in period $t = 1$. So this is a pure intertemporal economy. We make the point that the problem and its remedy has nothing to do with uncertainty. In particular, our rights are not trades on financial options. Indeed, in this example economy no securities will be traded, in equilibrium, and in this way we focus on the market for rights to trade in spot markets, only. Henceforth we drop all subscript s from the notation.

There are two types of agents, $H = 2$, both of which have an identical constant relative risk aversion (CRRA) utility function

$$u(c_1, c_2) = -\frac{1}{c_1} - \frac{1}{c_2}. \tag{26}$$

Each type consists of $\frac{1}{2}$ fraction of the population, i.e. $\alpha^h = \frac{1}{2}$. In addition, the discount factor is $\beta = 1$. The storage technology is given by $R = 1$. The endowment profiles of the agents are shown in Table 1 below. Note that an agent type 1 is well endowed with both goods in period $t = 0$ and vice versa for type 2. The first best allocation has both agents consuming 2 units of each good in every period. The first-best allocation thus suggests that agent 2 would like to move resources backwards in time from $t = 1$ to $t = 0$, i.e., borrow, and therefore will be constrained in the competitive collateral equilibrium. But borrowing requires collateral, and agent 2 is short of this as well. The equilibrium will have agent 2 borrowing nothing and only trading in spot markets. Agent 1 will be saving on its own to smooth consumption. Finally, in the equilibrium with rights to trade in spot markets, agent 1 will be paying more the higher is her storage/saving and this will be a force to do less. Likewise, agent 2 will be compensated for her participation in the rights markets, in effect providing resources for consumption in the first period, i.e., moving in the direction of the first best.

We summarize the equilibrium allocation in Table 1 featuring collateral k^h and consumption $c^h_{\ell s}$. See Appendix C for the derivation of the competitive equilibrium with the externality.

Table 1: Endowment profiles of the agents.

	endowments				equilibrium with the externality (ex)					equilibrium with rights to trade (op)				
	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}	k^h	c^h_{10}	c^h_{20}	c^h_{11}	c^h_{21}	k^h	c^h_{10}	c^h_{20}	c^h_{11}	c^h_{21}
$h = 1$	3	3	1	1	1.3595	2.6899	1.7756	1.3252	1.7756	1.1753	2.6073	1.8410	1.2970	1.6781
$h = 2$	1	1	3	3	0	1.3101	0.8649	2.6748	3.5839	0	1.3927	0.9837	2.7030	3.4972

There is no loss of generality to consider a solution with no security trading, i.e., $\theta^h_{\ell s} = 0$ for all h and for all ℓ. Agents do actively trade in spot markets. The price of good 2 in period $t = 0$ is $p^{ex}_0 = \left(\frac{4}{4-k^{ex}}\right)^2 = 2.2948$, and the market fundamental in period $t = 1$ is $z^{ex} = \frac{4}{4+k^{ex}} = 0.7463$, which implies that the spot price is $p\left(z^{ex}\right) = 0.5570$. Note that the collateral price at $t = 0$ is higher in the equilibrium with the externality than in the first-best, "fb", i.e., $p^{fb}_0 = 1 < p^{ex}_0 = 2.2948$. On the other hand, the spot price of good 2 in period $t = 1$ is lower in the equilibrium with the externality than in the first-best, i.e., $p\left(z^{fb}\right) = 1 > p\left(z^{ex}\right) = 0.5570$.

We will now turn to a corresponding competitive equilibrium with rights to trade in segregated exchanges (without the externality). There is *one active spot market*, $z^{op} = 0.7729$ ("op" stands for optimality), even though all spot markets are available in principle for trade. That is, in equilibrium, both types optimally choose to trade in the same spot market with specified market fundamental, $z^{op} = 0.7729$.

Table 2 presents equilibrium prices/fees of rights to trade in spot markets, that is $P_\Delta(z)$ not only for z^{op} but also other, different market fundamental levels z. Note again that the prices/fees of out-of-equilibrium (non-active) spot markets are available, but at such prices agents do not want to trade in them.

Table 2: Equilibrium prices of rights to trade in spot markets $P_\Delta(z)$. Bold numbers are equilibrium prices for actively traded spot markets.

	$z = 0.7479$	$\mathbf{z = 0.7729}$	$z = 0.7979$
$P_\Delta(z)$	0.4639	**0.5375**	0.6118

An agent type 1 is coming in with good 2 in storage, and therefore his discrepancy is positive. Type 1 pays for right to trade. On the other hand, an agent type 2's discrepancy is negative. Thus, with a positive equilibrium fee $P_\Delta(z^{op}) = 0.5375$, an agent type 2 must get paid for the access to the spot market. In particular, a constrained agent ($h = 2$) with $\Delta^2(z^{op}) = -0.6813$, is receiving a transfer of $-P_\Delta(z^{op})\Delta^2(z^{op}) = 0.3662$ in period $t = 0$ for being in the spot market $z^{op} = 0.7729$. Graphically, this shifts her budget line outward at $t = 0$ by $T = 0.3662$.[20]

Note that with lower aggregate saving, the price of good 2 in period $t = 0$ in this competitive equilibrium with segregated exchanges (without the externality) is lower ($p_0^{op} = 2.0073 < p_0^{ex} = 2.2948$) but the spot price of good 2 at $t = 1$ is higher ($p(z^{op}) = 0.5974 > p(z^{ex}) = 0.5570$), relative to the one in the competitive collateral equilibrium allocation (with

[20]Trading in rights to trade generates a redistribution of wealth and welfare in general equilibrium. The expected utility of an agent type 1 and type 2 in this competitive equilibrium with segregated exchanges (without the externality) are $U^1_{op} = -2.2936, U^2_{op} = -2.3905$, respectively. The expected utility of an agent type 1 and type 2 in the competitive collateral equilibrium allocation (with the externality) are $U^1_{ex} = -2.2527$ and $U^2_{ex} = -2.5724$, respectively. Thus if nothing else is done, internalizing the externality is beneficial to an agent type 2 (constrained agent) but harmful for an agent type 1. To induce welfare gains for all of agents, there must be lump sum transfers, as in the second welfare theorem, which we prove below.

the externality). That is, the price of good 2 varies less over time when the externality is internalized. In this sense we mitigate fluctuations. We do not move all the way to the first best.

The next example illustrates an economy with uncertainty where collateralized securities, θ^h_{1s}, are actively traded (cannot be substituted by spot trades). All agents are constrained, but at different states. In particular, an agent will be binding in a state where her endowment is large. This is because she would like to transfer a part of such a large amount of wealth backwards in time from $t = 1$ to $t = 0$ but cannot do so because of the collateral constraints.

Environment 2 (State Contingent Securities)**.** The economy in this example is similar to the one in Environment 1 with two periods, but there are two states, $S = 2$. There are two types of agents, $H = 2$, both of which have an identical constant relative risk aversion (CRRA) utility function as in (26). Each type consists of $\frac{1}{2}$ fraction of the population, i.e. $\alpha^h = \frac{1}{2}$. In addition, the discount factor $\beta = 1$. The storage technology is constant and given by $R_s = 1$ for $s = 1, 2$. The endowment profiles are presented in Table 3. Note, unlike the first example, that the agents are identical in endowments at $t = 0$. But agent type 1 has relatively more of both goods in state $s = 1$ than in state $s = 2$ and vice versa for agent type 2.

Table 3: Endowment profiles of the agents.

	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}	e^h_{12}	e^h_{22}
$h = 1$	2	2	3	3	1	1
$h = 2$	2	2	1	1	3	3

We will now solve for a competitive equilibrium with the externality. The detailed derivation is again omitted and presented in Appendix C. Collateralized securities $\theta^h_{\ell s}$ (or borrowing contracts) are $\theta^1_{11} = -\theta^1_{12} = -0.3042 = -\theta^2_{11} = \theta^2_{12}$. In words, an agent $h = 1$ issues (borrows) $\theta^1_{11} = -0.3042$ units of collateralized security paying good 1 at $s = 1$, and vice versa for an agent $h = 2$. We now turn to the competitive equilibrium with rights to trade. Each type of agent holds the same amount of collateral good $k^h = 0.4200 < 0.4603$, less than the one in competitive equilibrium with the externality. Collateralized securities (or borrowing contracts) are $\theta^1_{11} = -\theta^1_{12} = -0.2872 = -\theta^2_{11} = \theta^2_{12}$. Note that agents trade less securities relative to the equilibrium with the externality. This is because the agents save less and are issuing fewer securities. That is, the externality generates too much borrowing.

The following example presents an economy where it is possible to assign agents to different exchanges and have multiple segregated exchanges.

Environment 3 (Heterogeneous Borrowers and the Role of Mixtures)**.** There are three types of agents, two borrower types 2, 3 and one lender type 1. Each type consists of $\frac{1}{3}$ fraction of the population, i.e. $\alpha^h = \frac{1}{3}$, and all other aspects of the environment are as in Environment 1. The endowment profiles are given in Table 4 below.

Table 4: Endowment profiles of the agents.

Type of Agents	e^h_{10}	e^h_{20}	e^h_{11}	e^h_{21}
$h = 1$	4.26	11.5	0.5	0.5
$h = 2$	3.92	0.5	7	5
$h = 3$	4.32	0.5	5	7

Interestingly, there are now *two active spot markets*, $z = 0.6113$ and $z = 0.8132$ in the competitive equilibrium with segregated exchanges. The spot market $z = 0.6113$ consists of some fraction of agents type 1 (19.69 percent), and all of agents type 3 (a constrained type). On the other hand, the spot market $z = 0.8132$ consists of some residual fraction of agents type 1 (80.31 percent), and all of agents type 2 (a constrained type). We use the term mixtures to refer to the fact that agent 1 is allocated to two active markets in some nontrivial proportions.

Equilibrium fees of rights to trade in spot markets, including the fees of inactive (out-of equilibrium) spot markets are summarized in Table 5 below.

Table 5: Equilibrium fees of rights to trade in spot markets. The bold numbers are (actively traded) equilibrium prices.

	$z = 0.6088$	$\mathbf{z = 0.6113}$	$z = 0.6138$	$z = 0.8088$	$\mathbf{z = 0.8132}$	$z = 0.8138$
$P_\Delta(z)$	0.9119	**0.9348**	0.9589	2.2339	**2.2537**	2.2564

It is socially optimal to compensate constrained agents with positive transfers at period $t = 0$, to try to move back toward the first best, i.e., alleviate borrowing constraints. In this example the number of active segregated spot markets is equal to the number of constrained types, to allow this to happen.

In the competitive equilibrium with segregated exchanges (without the externality), the discrepancy from the market fundamental of both constrained types are negative, i.e., $\Delta^2 = -2.9340$ and $\Delta^3 = -0.7209$. With positive equilibrium price of the discrepancy, agents type 2 and agents type 3 receive transfers from rights to trade fees $P_\Delta(z)\,\Delta^h(z)$ of 6.6122 units of good 1 in period $t = 0$ and 0.6739, respectively. Agents type 1 buy a lottery which is actuarially fair and thus pay fees paid in proportion to the relative number of its type assigned to each exchange. Agents type 1 would like to buy into the higher spot market, which is $z = 0.8132$ in this case, where good 2 is more valuable because with (endogenous) saving she will end up with more of good 2 than good 1 in period $t = 1$ but such a deterministic choice is not affordable.

2.1.7 The Existence of Competitive Equilibrium with Segregated Exchanges and the Welfare Theorems

The formal notation for competitive equilibrium with mixtures is in Appendix A. Suffice it to note here that as in the classical general equilibrium model, the economy is a well-defined convex economy, i.e., the commodity space is Euclidean, the consumption set is compact and convex, the utility function is linear. As a result, the first and second welfare theorems hold, and a competitive equilibrium exists.

The standard contradiction argument is used to prove the first welfare theorem below. We assume that there is no local satiation point in the consumption set.

Theorem 1. *With local nonsatiation of preferences, a competitive equilibrium with segregated exchanges allocation is constrained optimal.*

Proof. See Appendix B. □

The Second Welfare theorem states that any constrained optimal allocation, corresponding to strictly positive Pareto weights, can be supported as a competitive equilibrium with segregated exchanges with transfers. The standard approach applies here. Essentially, decentralized prices are coming from the Lagrange multipliers for the resource constraints of the planning problem.

Theorem 2. *With locally non-satiated utility functions that satisfy the Inada conditions, any constrained optimal allocation corresponding with strictly positive Pareto weights $\lambda^h > 0, \forall h$ can be supported as a competitive equilibrium with segregated exchanges and with lump sum transfers.*

Proof. See Appendix B. $\square$

We use Negishi's mapping method (Negishi, 1960) to prove the existence of competitive equilibrium with segregated exchanges.

Theorem 3. *With locally non-satiated utility functions that satisfy the Inada conditions and positive endowments, a competitive equilibrium with segregated exchanges exists.*

Proof. See Appendix C. $\square$

2.2 An Exogenous Incomplete Markets Economy

As before, consider again an economy with two periods, $t = 0, 1$. There are S possible states of nature in the second period $t = 1$, i.e., $s = 1, \ldots, S$, each of which occurs with probability π_s such that $\sum_s \pi_s = 1$. There are 2 goods, labeled good 1 and good 2, in each date and in each state. Because the endowment profiles are the same as specified in the collateral economy discussed above, we omit notational details in this section for brevity. We do stress that now there is no saving as our focus is no longer on collateral.

There are J securities available for purchase or sale in the first period, $t = 0$. Let $\mathbf{D} = [D_{js}]$ be the payoff matrix of those assets where D_{js} be the payoff of asset j in units of good 1 (the numeraire good) in state $s = 1, 2, \ldots, S$ in the second period $t = 1$. Here we do not include securities paying in good 2 as there must be spot markets anyway. Let θ_j^h denote the amount of the j^{th} security acquired by an agent of type h at $t = 0$ with $\boldsymbol{\theta}^h \equiv \left[\theta_j^h\right]_j$, and Q_j denote the price of security j with $\mathbf{Q} \equiv [Q_j]_j$. An exogenous incomplete markets assumption specifies that $\mathbf{D}$ is not full rank; that is, $J < S$. This is crucial for the model of this section. In particular, spot trades become essential but create an externality.

As before, let $\tau_{\ell s}^h$ denote spot trade amount of good $\ell = 1, 2$ in spot markets in state s acquired by an agent of type h. Let p_0 and p_s denote the prices of good 2 in units of good 1 in period $t = 0$ and at state s in period $t = 1$, respectively. Here spot market trade is generically essential given the incomplete security structure.

The preferences of an agent of type h are represented by the utility function $u^h\left(c_1^h, c_2^h\right)$, and the discounted expected utility of h is defined by:

$$U^h\left(\mathbf{c}^h\right) \equiv u^h\left(c_{10}^h, c_{20}^h\right) + \beta \sum_{s=1}^{S} \pi_s u^h\left(c_{1s}^h, c_{2s}^h\right), \tag{27}$$

where β is the discount factor. Note that there would be no externalities if preferences were identically homothetic, as spot prices are determined by ratio of aggregate endowment only, which, with no storage, no one can influence. So we assume otherwise; that is, preferences are now not identically homothetic.

Definition 4 (Competitive Equilibrium with Exogenous Incomplete Market)**.** A competitive equilibrium is a specification of prices $(p_0, \mathbf{Q}, \mathbf{p})$, and an allocation $\left(\mathbf{c}_0^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)_h$ such that

- for any agent type h as a price taker, $\left(\mathbf{c}_0^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h\right)$ solves

$$\max_{\mathbf{c}_0^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h} u^h\left(c_{10}^h, c_{20}^h\right) + \beta \sum_s \pi_s u^h\left(e_{1s}^h + \sum_j D_{js}\theta_j^h + \tau_{1s}^h, e_{2s}^h + \tau_{2s}^h\right) \tag{28}$$

  subject to the budget constraints in the first period

$$c_{10}^h + p_0 c_{20}^h + \sum_{j=1}^{J} Q_j \theta_j^h \quad \leq \quad e_{10}^h + p_0 e_{20}^h, \tag{29}$$

  and the spot budget constraint in state s

$$\tau_{1s}^h + p_s \tau_{2s}^h \quad = \quad 0, \text{ for } s = 1, \ldots, S; \tag{30}$$

- markets clear for good $\ell = 1, 2$ at $t = 0$, for θ_j^h for all $j = 1, \ldots, J$, and for spot trade

$\tau_{\ell s}^{h}$ in state s, respectively:

$$\sum_h \alpha^h c_{\ell 0}^h = \sum_h \alpha^h e_{\ell 0}^h, \forall \ell = 1, 2, \tag{31}$$

$$\sum_h \alpha^h \theta_j^h = 0, \forall j, \tag{32}$$

$$\sum_h \alpha^h \tau_{\ell s}^h = 0, \forall s; \ell = 1, 2. \tag{33}$$

The key constraints that generate the externality in this problem are the spot-budget constraints (30) for an agent of type h. Note that the spot price p_s is determined by pretrade position of endowments and securities where endowments are exogenous but securities are endogenous, and we write this as $p_s = p_s(\boldsymbol{\theta}, \mathbf{e})$. As in Geanakoplos and Polemarchakis (1986), the dependency generates an indirect price effect from security reallocations. This indirect effect then produces an externality when the security markets are incomplete[21].

2.2.1 Competitive Equilibrium with Segregated Exchanges

Each security exchange in this case must deal with S spot markets as a bundle. This is due to the restriction of the incompleteness of the markets; we cannot separately write down contingent contracts for each state. More formally, each security exchange trades the vector bundle of the rights to trade in the spot markets at particular vector of prices $\mathbf{p} = (p_s)_{s=1}^S$, and for an agent type h the right to trade in the spot markets at the same particular vector $\mathbf{p}$ is defined as a vector $\boldsymbol{\Delta}^h(\mathbf{p}) = \left[\Delta_s^h(\mathbf{p})\right]_{s=1}^S$.

More specifically, type h's right to trade in exchange $\mathbf{p}$ for particular state s is $\Delta_s^h(\mathbf{p})$ and is defined as the excess demand for good 1 of an agent type h in spot markets in state s, i.e., $\Delta_s^h(p_1, \ldots, p_s, \ldots, p_S) = \tau_{1s}^{h*}\left(\mathbf{e}_s^h, \boldsymbol{\theta}^h, p_s\right)$, which is the solution to the following utility maximization:

$$\left(\tau_{1s}^{h*}\left(\mathbf{e}_s^h, \boldsymbol{\theta}^h, p_s\right), \tau_{2s}^{h*}\left(\mathbf{e}_s^h, \boldsymbol{\theta}^h, p_s\right)\right) = \underset{\tau_{1s}^h, \tau_{2s}^h}{\operatorname{argmax}}\, u^h\left(e_{1s}^h + \sum_{j=1}^J D_{js}\theta_j^h + \tau_1^h, e_{2s}^h + \tau_2^h\right) \tag{34}$$

subject to the spot-budget constraints (30).

[21]When the security markets are complete, these indirect price effects are canceling each other out, and as a result, the competitive equilibrium with exogenous security markets is (constrained) efficient as expected. This statement is formally proved in a proposition in Appendix C.

As with our initial formulation in the collateral economy, we allow each agent to choose one but only one exchange indexed by $\mathbf{p} = (p_1, \ldots, p_s, \ldots, p_S)$, where it can trade good 2 at spot price p_s in each state s. An agent's choice ranges over all potential price vector $\mathbf{p}$. More formally, let $\delta^h(\mathbf{p}) \in \{0, 1\}$, that is $\delta^h(\mathbf{p}) = 0$ or $\delta^h(\mathbf{p}) = 1$, denote an agent type h's discrete choice of an exchange indexed by $\mathbf{p}$.

At $t = 0$, each agent type h is choosing both the spot market prices it desires and the security trades jointly. Thus the consistency constraint for each exchange $\mathbf{p}$ that is active then can be derived from the spot market clearing condition, which can be rewritten as follows.

$$\sum_h \alpha^h \delta^h(\mathbf{p}) \Delta_s^h(\mathbf{p}) = 0, \forall s, \mathbf{p}. \tag{35}$$

Each consistency constraint ensures that the composition of agent types in an exchange $\mathbf{p} = (p_1, \ldots, p_s, \ldots, p_S)$ is such that the spot market clearing prices in each state s is p_s. Notice that these constraints are for active exchanges, not for every possible $\mathbf{p}$.

Let $Q_j(\mathbf{p})$ denote the price of security j traded in exchange $\mathbf{p}$ with $\mathbf{Q} \equiv [Q_j(\mathbf{p})]_{j,\mathbf{p}}$, and $P_\Delta(\mathbf{p}, s)$ denote the market price of rights to trade in exchange $\mathbf{p}$ for state-s spot market, $\Delta_s^h(\mathbf{p})$ with $\mathbf{P}_\Delta \equiv [P_\Delta(\mathbf{p}, s)]_{s,\mathbf{p}}$.

Definition 5. A competitive equilibrium with segregated exchanges is a specification of allocation $\left(x^h(\mathbf{p})\right)_{h,\mathbf{p}} \equiv \left(\delta^h(\mathbf{p}), \mathbf{c}_0^h(\mathbf{p}), \boldsymbol{\theta}^h(\mathbf{p}), \boldsymbol{\tau}^h(\mathbf{p}), \boldsymbol{\Delta}^h(\mathbf{p})\right)_{h,\mathbf{p}}$ and prices $(p_0, \mathbf{Q}, \mathbf{p}, \mathbf{P}_\Delta)$ such that

- for any agent type h as a price taker, $\left[x^h(\mathbf{p})\right]_\mathbf{p}$ solves

$$\max_{\left[x^h(\mathbf{p})\right]_\mathbf{p}} \sum_\mathbf{p} \delta^h(\mathbf{p}) \left[u\left(c_{10}^h(\mathbf{p}), c_{20}^h(\mathbf{p})\right) + \sum_s \pi_s u\left(e_{1s}^h + \sum_j D_{js}\theta_j^h(\mathbf{p}) + \tau_{1s}^h(\mathbf{p}), e_{2s}^h + \tau_{2s}^h(\mathbf{p})\right)\right]$$

  subject to the budget constraints in the first period

$$\sum_\mathbf{p} \delta^h(\mathbf{p}) \left[c_{10}^h(\mathbf{p}) + p_0 c_{20}^h(\mathbf{p}) + \sum_j Q_j(\mathbf{p}) \theta_j^h(\mathbf{p}) + \sum_s P_\Delta(\mathbf{p}, s) \Delta_s^h(\mathbf{p})\right] \leq e_{10}^h + p_0 e_{20}^h,$$

  and the spot-budget constraint in state s

$$\sum_\mathbf{p} \delta^h(\mathbf{p}) \left[\tau_{1s}^h(\mathbf{p}) + p_s \tau_{2s}^h(\mathbf{p})\right] = 0, \forall s,$$

- markets clear for good ℓ in $t=0$, for securities j paying good 1, for good ℓ in state s, and for rights to trade in exchange $\mathbf{p}$ for state s, respectively,

$$\sum_h \sum_{\mathbf{p}} \delta^h(\mathbf{p})\,\alpha^h c_{\ell 0}^h(\mathbf{p}) = \sum_h \alpha^h e_{\ell 0}^h, \forall \ell = 1,2,$$

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \theta_j^h(\mathbf{p}) = 0, \forall j; \mathbf{p},$$

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \tau_{\ell s}^h(\mathbf{p}) = 0, \forall s; \mathbf{p}; \ell = 1,2,$$

$$\sum_h \delta^h(\mathbf{p})\,\alpha^h \Delta_s^h(\mathbf{p}) = 0, \forall s; \mathbf{p}.$$

As in the collateral economy, markets for the rights to trade can remove the externality. The formal argument[22] is quite similar to the collateral economy in Section 2.1.

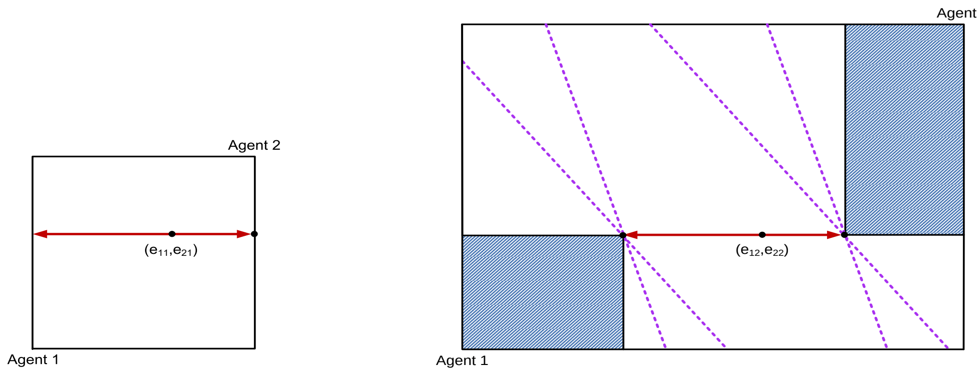
Of course, one might wonder if our method solves the externality problem by simply completing the markets? By allowing agents to choose markets with pre-specified spot prices in each state s, we effectively create state-contingent transfers of wealth at least to some degree. But is it enough to achieve the first best allocation? The answer is generally, no. Exogenous incomplete markets and the positivity of spot prices still restrict how much wealth transfers we can make in each state. Technically, the feasible set with incomplete markets and rights to trade is generically a strict subset of the feasible set with the complete markets.

See Figure 1 for an illustrative example. This example assumes the stereotypical debt contract that pays the same amount of good 1 in each two states. However, in state $s=2$, there is more of good 1 and good 2 overall. Then, no matter what the price ratio p_s in state $s=2$, certain regions cannot be reached. The main point is that the scarcity in state $s=1$ can affect the feasibility in state $s=2$ because the markets are incomplete.

3 Mapping Economies into A Generalized Framework

The key point is that our market-based solution concept is applicable to many economies in which agents face a friction that generates constraints containing spot market prices. So

[22] In Kilenthong and Townsend (2014a), we prove the welfare theorems for a general model that fits each of the two leading examples as special cases.



(a) A feasible set in state $s = 1$ when only available security pays the same amount of good 1 in both states.

(b) A feasible set in state $s = 2$ when only available security pays the same amount of good 1 in both states. The shaded areas are not feasible.

Figure 1: Feasible Sets in state $s = 1$ and $s = 2$ when markets are incomplete.

here we proceed in reverse and lay out a generalized framework much as in Prescott and Townsend (1984b), then show several well known environments in the literature are special cases.

There are at least 6 prototype economies that fit into our framework. These include collateral economy as in Section 2.1, exogenous incomplete markets as in Section 2.2, fire sales economy (Lorenzoni, 2008), liquidity constrained economy (Hart and Zingales, 2013), moral hazard with retrading (Acemoglu and Simsek, 2012; Kilenthong and Townsend, 2011), and hidden information with retrading (Diamond and Dybvig, 1983). For brevity, this section presents only the key constraints in each economy without describing the entire environment of the model nor the equilibrium with the new markets for rights to trade. See Kilenthong and Townsend (2014a) for the general notation that encompasses all these examples.

The general form of those constraints generating pecuniary externalities is actually easy to write. It is denoted by $C^h\left(\mathbf{c}^h, \boldsymbol{\theta}^h, \boldsymbol{\tau}^h, \mathbf{y}^h, \mathbf{p}\right)$, where $\mathbf{y}^h$ is the vector of inputs and outputs for an associated production technology and $\mathbf{p}$ is the vector of spot prices. The notation of consumption $\mathbf{c}^h$, securities $\boldsymbol{\theta}^h$, and trades $\boldsymbol{\tau}^h$ is as before. The key feature is the introduction of price vector $\mathbf{p}$.

For the collateral economy, the key friction can be written as the following constraints:

$$C_s^h\left(\theta_{\ell s}^h, y_{2s}^h, p_s\right) \equiv p_s y_{2s}^h + \theta_{1s}^h + p_s \theta_{2s}^h \geq 0, \forall s, h, \tag{36}$$

which are equivalent to the collateral constraints (1) with $y_{2s}^h = R_s k^h$.

For exogenous incomplete markets, the key constraints are

$$C_s^h\left(\tau_{\ell s}^h, p_s\right) \equiv \tau_{1s}^h + p_s \tau_{2s}^h = 0, \forall s, h, \tag{37}$$

which is identical to the spot budget constraints (30).

For the fire sales economy of Lorenzoni (2008), the key constraint that causes an inefficiency is the following no-default condition:

$$C_{ed1}^e\left(y_{n00}^e, \theta_{1s}^e, \theta_{2s}^e, p_s\right) = \left(\eta a_s + \max\left\{p_s - \gamma, 0\right\}\right) y_{n00}^e + \theta_{1s}^e + \theta_{2s}^e \geq 0, \forall s = 1, 2. \tag{38}$$

Constraints (38) imply that the entrepreneur is better off not defaulting at state $s = 1, 2$ in period $t = 1$. Here y_{n00}^e is the capital input in period $t = 0$, a_s is the productivity in state s, γ is the cost to repair a unit of capital in period $t = 0$, θ_{ts}^e are securities paying in unit of consumption goods at state s in period $t = 1, 2$, and $1 - \eta \in (0, 1)$ is the fraction of the firm's current profit that the entrepreneur could keep if he decided to default. For more detail see Kilenthong and Townsend (2014a). The key point is that this constraint depends on equilibrium prices p_s which in turn are determined by collective ex-ante choices of the agents.

The next economy is the liquidity constrained economy of Hart and Zingales (2011), where there are two types of agents, namely doctors d and builders b. The key constraints that cause an inefficiency are the following spot market constraints in period $t = 1, 2$, respectively:

$$C_1^h\left(\tau_{f1}^h, \tau_b^h, p_b\right) = \tau_{f1}^h + p_b \tau_b^h = 0, \forall h = b, d, \tag{39}$$

$$C_2^h\left(\tau_{f2}^h, \tau_d^h, p_d\right) = \tau_{f2}^h + p_d \tau_d^h = 0, \forall h = b, d. \tag{40}$$

These constraints state that each agent can trade building or doctor service when it is available with their storage claim (as the numeraire goods in each period) received at that time, which is the liquidity in the model. Note that this is an incomplete market model. Here τ_{ft}^h, τ_b^h, and τ_d^h are the storage outcome at period t, spot trade for building service, and spot trade for doctor service, respectively, and p_b and p_d are the spot-market-clearing prices of building and doctor services in period $t = 1$ and $t = 2$, respectively.

For the moral hazard with retrading economy of (Acemoglu and Simsek, 2012; Kilenthong and Townsend, 2011), the key constraints that cause an inefficiency are the following incentive compatibility constraints (IC):

$$C_{1,a,a'}\left(\mathbf{c},p\right)=\sum_{\mathbf{q}}u\left(c_1(\mathbf{q},a),c_2(\mathbf{q},a),a\right)f(\mathbf{q}|a)-\sum_{\mathbf{q}}v\left(c_1(\mathbf{q},a),c_2(\mathbf{q},a),a',p\right)f(\mathbf{q}|a')\geq 0,\forall a,a'. \quad (41)$$

The incentive compatibility constraints ensure that the agent takes the recommended action a and so $a'=a$. Mor especifically, here a is the recommended action, a' is an alternative action, $\mathbf{q}=(q_1,q_2)$ is the output vector of good 1 and good 2, $f(\mathbf{q}|a)$ is the probability production technology, $c_\ell(\mathbf{q},a)$ is the optimal consumption of good ℓ condition on realized output $\mathbf{q}$ and recommended action a, and the value function under the alternative action a' is

$$v\left(c_1(\mathbf{q},a),c_2(\mathbf{q},a),a',p\right)=\max_{\tau_1,\tau_2}u\left(c_1(\mathbf{q},a)+\tau_1,c_2(\mathbf{q},a)+\tau_2,a'\right) \quad (42)$$

subject to the spot budget constraint:

$$\tau_1+p\tau_2=0, \quad (43)$$

taking the spot price p as given.

The last prototype economy is the hidden information with retrading economy of (Diamond and Dybvig, 1983). The key constraints that causes an inefficiency are the following truth-telling constraints (IC):

$$C_{1,\eta,\eta'}\left(\mathbf{c},p\right)=u\left(c_1\left(\eta\right),c_2\left(\eta\right),\eta\right)-v\left(c_1\left(\eta'\right),c_2\left(\eta'\right),\eta,p\right)\geq 0,\forall\eta,\eta', \quad (44)$$

The truth-telling constraints ensure that the agent report the true shock, and so $\eta'=\eta$. Here η is the true shock/state, η' is the reported shock, $c_\ell\left(\eta'\right)$ is consumption under the true shock, $c_\ell\left(\eta'\right)$ is consumption conditional on the reported shock η', and the value function under the reported shock η' is

$$v\left(c_1\left(\eta'\right),c_2\left(\eta'\right),\eta,p\right)=\max_{\tau_1,\tau_2}u\left(c_1\left(\eta'\right)+\tau_1,c_2\left(\eta'\right)+\tau_2,\eta\right) \quad (45)$$

subject to the spot budget constraint:

$$\tau_1+p\tau_2=0, \quad (46)$$

taking the spot price p as given.

To sum up, agents in each of these economies face a set of frictions that generates constraints containing spot market prices similar to the ones in the collateral economy and the exogenous incomplete markets economy, which are presented in Section 2.1 and 2.2, respectively. Our new concept of the markets for rights to trade in segregated exchanges can be applied in the same way to remove an externality in these economies. Formally, let $\Delta_s^h(\mathbf{p})$ denote the amount of the rights to trade in a particular security exchange in state s with a specified price p_s. That is, in order to be eligible to trade in this exchange $\mathbf{p} \equiv (p_1, p_s, \ldots, p_s, \ldots, p_S)$, an agent of type h must hold the rights to trade $\Delta_s^h(\mathbf{p})$ equals to its excess demand for good 1 in that spot market $d_s^h\left(\mathbf{e}_s^h, \boldsymbol{\theta}^h, \mathbf{y}^h, p_s\right)$. These rights to trade at $\mathbf{p}$ have its own unit of account market prices, $P_\Delta(\mathbf{p}, s)$. See Kilenthong and Townsend (2014a) for more details.

4 Discussion on Implementation

Virtually all the technological ingredients of what would be needed to implement our solution are available in actual securities markets, including financial markets that are currently susceptible to fire sales. Below we try to be as specific as possible in a particular application, with parenthetical remarks to more general considerations.

We begin with the Generalized Collateral Financing Repo. This is a financial platform organized by the Fixed Income Clearing Corporation (FICC), designed to allow securities dealers to buy/sell, borrow/lend securities and cash among themselves and to do the netting. It originated in 1998 and came from a merger of two separate platforms. (This makes the point that new financial platforms can be created, here with a merger but divisions and segmentation are also possible). US Treasuries and Fannie Mae & Fredie Mac MBS are the most common securities traded among other government issued or backed financial instruments. (Here we imagine our model environment which is written for agent types with utility over consumption goods is actually an economy with traders/dealers with indirect preferences over cash and securities having to do with their customer demands, which we do not model.) Securities are held, maintained, and registered on electronic book entry systems of FICC, the Clearing Banks, the Federal Reserve and the US Treasury. Some direct transfers of securities are made through Fedwire Securities Service with payment in Fedwire Funds. It is not possible to transfer legal ownership of securities outside of these utilities. Our assumptions of exclusivity do not require new technology.

A typical repossession or repo transaction is a sale of a security with an agreed upon price, ex ante, at which it will be bought back. These are like borrowing and lending transactions, in which a borrower in need of cash from an investor gives up securities to be held in escrow until the loan is repaid. In the GCF repo market securities are placed at the principal-plus-interest loan amount only (the FICC acts as a guarantor). The difference between the sale price and the higher repurchase price is the interest rate. We emphasize that the repurchase price is part of the contract. All initial contracts, agreed upon transactions, and collateral are recorded and trades executed with the two major clearing banks.

Repo markets are potentially subject to fire sale risks. There is post-default risk. When a dealer borrower defaults on the repo, its investors receive the securities posted as collateral. (Please recall that the securities in our model may well default. No default was assumed without loss of generality as any security which does default, with collateral passing to the investor is equivalent with another, that we use explicitly in the analysis, which does not default. We model the sale of collateral by investors as if in a competitive market.)[23] We emphasize here this default and sale aspect. Related, a borrower may wish to get some of the securities used as collateral back during the trading cycle and in that case the clearing bank determines the cash value at current market prices, at that point in time. Our point here is that the clearing banks could value and unwind collateral in these instances at pre specified prices, agreed at the time when the trade is entered into. Some exchanges have liquidation auctions to value assets, for example, those of a defaulting member of a clearing house, with the auction restricted to members as named players. Our point here is that one could imagine something like index credit default swaps traded ex ante in market exchanges as insurance against default, and also that traders would pay or be paid to participate in this bidding. Payment would be debited/credited from payment accounts in the contract

[23]There is also a pre-default risk of fire sales which is reminiscent of Lorenzoni's tale of Thailand. Stressed dealers face difficulties raising funds in tri-party repo when investors worry about the counterparty risk, causing them to de-lever and sell securities.

period, so there would be no time inconsistency problem on that dimension.

In practice trades in the GCF repo market are among dealers and are placed with inter-dealer brokers (IDBs). In 2012 there were roughly 120 dealers and 5 IDB platforms. All dealers must be approved by the FICC. Trades of a dealer with an IDB are conducted by voice, allowing negotiation or electronically, allowing anonymous trading with platform/market determined prices. Though buyers are matched with sellers, the bilateral nature of the market can fade away; that is, in many models, in the limit with a large number of traders, the outcome is Walrasian as we assume here from the get go. See in particular Townsend (1983) and Kilenthong and Qin (2014) for an intermediary that could be interpreted as an IDB announcing prices and trying to attract trades. Our point here, again, is that one could imagine financial platforms which post prices in advance for unwinding collateral and charge fees or provide compensation for participation.

Again exclusivity is not a problem given current technology. The current market structure has exclusivity embedded, as is clear in this discussion of broker dealers, FICC, clearing banks, and the Fed for trade, clearing, and settlement. More generally, the Fed has a list of authorized broker dealers for the OTC treasury market. Mutual funds and other investors are not allowed to deal directly on Fedwire funds and Fedwire securities and go through broker dealers. More generally, some market exchanges are said to restrict access to high frequency traders. Or in yet another example, exchange traded funds name a restricted set of Authorized Participants who are allowed to deal with the sponsor of the fund. Finally, as a consequence of Dodd Frank regulation, market participants in CDS index contracts are required to trade in a Swap Execution Facility and to clear in a CCP platform. These platforms charge fees and commission, based typically on volume.

To sum up, the technology exists to register securities, to monitor trades, to enforce agreements, to auction assets, and to hedge default risk and security price movements ex ante.

5 Conclusion

Here we draw on the insights of Coase (1960) and Lindahl (1958), extend the commodity space as in Arrow (1969), overcome some conceptual and technical hurdles, and show how the appropriate set of markets can eliminate fire sale externalities and the inefficiency of incomplete security markets. Our solution concept extends to many other well known environments in the literature. By its nature, a pecuniary externality has to do with the impact of prices in constraints beyond the role of prices in budget constraints, as happens in many models. The solution can be put rather simply: create segregated market exchanges which specify prices in advance (but with the same prices that also clear active markets ex post) and price the right to trade in these markets so that participant types pay, or are compensated, consistent with the market exchange they choose and that type's excess demand contribution to the price in that exchange.

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A Mixture Representation of the New Markets

To deal with the non-convexity problem generated by the collateral constraints, we now use a probability measure, which is a mixture at the aggregate and a lottery at the individual level. That is, we now suppose it is possible to assign agents to different exchanges even in state s. Security trades are also bundled into this potentially random assignment.

More formally, with a continuum of agents, let $x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ be the fraction of agents type h assigned to a bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$. At the individual level, for each agent type h, let $x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) \geq 0$ denote a probability measure on $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$. In other words, $x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ is the probability of receiving period $t = 0$ consumption

$\mathbf{c}_0 \equiv (c_{10}, c_{20})$, collateral k, securities $\boldsymbol{\theta} = [\theta_{\ell s}]_{\ell,s}$, spot trade $\boldsymbol{\tau} = [\tau_{\ell s}]_{\ell,s}$, and being in exchanges indexed by $\mathbf{z} \equiv [z_s]_s$ with rights to trade $\boldsymbol{\Delta} \equiv [\Delta_s(z_s)]_s$.

All securities contracts are entered into ex-ante at $t = 0$, and spot trades and the valuation of collateral take place at spot price $p(z_s)$. Unlike the previous discrete choice x^h notation in Section 2.1.5, it is not necessary to index all the objects in the commodity vector by $\mathbf{z}$. This is because a probability of objects conditioned on $\mathbf{z}$ times the marginal probability of $\mathbf{z}$ can be rewritten as a joint probability with $\mathbf{z}$ is an object in the commodity vector. But this still allows many of the objects chosen to be degenerate, as in the example of Environment 3 earlier, where only agent type 1 chooses a lottery on $\mathbf{z}$ and securities conditioned on that draw of $\mathbf{z}$ are degenerate.

As a probability measure, a lottery of an agent type h satisfies

$$\sum_{\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z},\boldsymbol{\Delta}} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) = 1. \tag{47}$$

Each bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ will be feasible only if the collateral and security assignments satisfy the collateral constraints (13), and there is a relationship among $\mathbf{z}$ and the $\Delta_s(z_s)$, $s = 1, 2, \ldots, S$ for each z_s, namely equation (11).

Accordingly, we impose the following condition on a probability measure

$$\begin{aligned} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) &\geq 0 \text{ if } (\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) \text{ satisfies (11) and (13),} \\ &= 0 \text{ if otherwise.} \end{aligned} \tag{48}$$

More formally, the consumption possibility set[24] of an agent type h is defined by

$$X^h = \left\{ \mathbf{x}^h \in \mathbb{R}^n_+ : \sum_{\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z},\boldsymbol{\Delta}} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) = 1, \text{ and (48) holds} \right\}. \tag{49}$$

Note that X^h is compact and convex. In addition, the non-emptiness of X^h is guaranteed by assigning mass one to each agent's endowment, i.e., no trade is a feasible option.

[24] More formally, with all choice objects gridded up as an approximation, the commodity space L is assumed to be a finite n-dimensional linear space. The limiting arguments under weak-topology used in Prescott and Townsend (1984a) can be applied to establish the results if L is not finite.

Pareto Program with Segregated Exchanges

We now can write down the programming problem for the determination of ex ante Pareto optimal allocations. Instead of maximizing type 1 utility subject to parametric reservation utilities for the other type, as Program 1, here we just maximize the λ-weighted sum of expected utilities. But there are utilities and λ's which make them equivalent. Of course we need all the resource constraints on consumption, securities, and transfers. In sum we have

Program 2.

$$\max_{\left(\mathbf{x}^h \in X^h\right)_h} \sum_h \lambda^h \alpha^h \sum_{\mathbf{c}_0, k, \theta, \mathbf{z}, \boldsymbol{\Delta}} x^h \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) V^h \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) \tag{50}$$

where $V^h \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) = u\left(c_{10}^h, c_{20}^h\right) + \beta \sum_s \pi_s u\left(e_{1s}^h + \theta_{1s} + \tau_{1s}, e_{2s}^h + R_s k + \theta_{2s} + \tau_{2s}\right)$ is the expected utility value derived from a bundle $\left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right)$ by an agent type h. Objective function (50) is subject to resource constraints for good 1 at $t = 0$, good 2 at $t = 0$, securities $\theta_{\ell s}$, spot trades $\tau_{\ell s}$, and the rights to trade $\Delta_s\left(z_s\right)$, respectively:

$$\sum_h \alpha^h \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}} x^h \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) c_{10} = \sum_h \alpha^h e_{10}^h, \tag{51}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}} x^h \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) \left[c_{20} + k\right] = \sum_h \alpha^h e_{20}^h, \tag{52}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}_{-s}, \boldsymbol{\Delta}} x^h \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) \theta_{\ell s} = 0, \forall s; z_s; \ell = 1, 2, \tag{53}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}_{-s}, \boldsymbol{\Delta}} x^h \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) \tau_{\ell s} = 0, \forall s; z_s; \ell = 1, 2, \tag{54}$$

$$\sum_h \alpha^h \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}_{-s}, \boldsymbol{\Delta}} x^h \left(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}\right) \Delta_s\left(z_s\right) = 0, \forall s; z_s; \ell = 1, 2. \tag{55}$$

Note again that the resource constraints (51) and (52) imply that all agent types can trade good 1 and good 2 at $t = 0$ regardless of their choice of exchange z_s. On the other hand, constraints (53)-(55) imply that each member of an exchange z_s can trade ex-ante securities, spot trades, and rights to trade with other members in the same exchange only.

The optimal condition with respect to any particular $x^h \left(\mathbf{c}_0, k, \theta, \tau, \mathbf{z}, \boldsymbol{\Delta}\right)$ for Pareto Program 2 is

$$\lambda^h V^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) \leq \widetilde{P}_{10} c_{10} + \widetilde{P}_{20} c_{20} + \widetilde{P}_{20} k + + \sum_{\ell,s} \widetilde{Q}_\ell(z_s, s)\, \theta_{\ell s}$$
$$+ \sum_{\ell,s} \widetilde{P}_\ell(z_s, s)\, \tau_{\ell s} + \sum_s \widetilde{P}_{\boldsymbol{\Delta}}(z_s, s)\, \Delta_s + \widetilde{P}^h_l \tag{56}$$

where $\left\{\widetilde{P}_{\ell 0}, \widetilde{Q}_\ell(z_s, s), \widetilde{P}_\ell(z_s, s), \widetilde{P}_\Delta(z_s, s)\right\}$ are the shadow prices of the resource constraints, $\tilde{P}^h_l$ is the Langrange multiplier for the probability constraint (49), and the inequality holds with equality if $x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) > 0$. Of course, the choice objects $x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ can be zero, no mass, if the marginal gain is less than the costs at shadow prices. One might note in particular that constraint (55) delivers shadow prices not only for each s but also for every possible z_s conditioned on state s. All of them are satisfied at equality, with the inactive exchanges trivially since there is no mass there (in the underlying commodity space excess demands are zero for all types in inactive exchanges).

Competitive Equilibrium with Segregated Exchanges in The Mixture Representation

To decentralize, let $P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ be the price for a commodity point $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$. However we can already guess from the planning problem that, apart from a normalization to express in terms of the numeriare good 1, in the equilibrium

$$P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) = c_{10} + p_0 c_{20} + p_0 k + \sum_{\ell,s} Q_\ell(z_s, s)\, \theta_{\ell s} + \sum_{\ell,s} P_\ell(z_s, s)\, \tau_{\ell s} + \sum_s P_\Delta(z_s, s)\, \Delta_s. \tag{57}$$

Thus we have a representation of prices on the objects separately; that is, on consumption goods at $t = 0$, security purchases or issues, spot market consumption, and the market rights. For convenience, though we retain the short-hand P notation. Thus, for consumers: each agent type h chooses $\mathbf{x}^h$ in period $t = 0$ to maximize its expected utility:

$$\max_{\mathbf{x}^h} \sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})\, V^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) \tag{58}$$

subject to $\mathbf{x}^h \in X^h$, and period $t = 0$ budget constraint

$$\sum_{\mathbf{c}_0, k, \theta, \tau, \mathbf{z}, \boldsymbol{\Delta}} P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})\, x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) \quad \leq \quad e^h_{10} + p_0 e^h_{20}, \tag{59}$$

taking price of good 2 at $t = 0$, p_0, and prices of lottery, $P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ as given.

As with Arrow's original paper, we need something on the production side, since a type's action enters into the clearing constraints for the market rights $\Delta_s(z_s)$ (the consistency constraints) and hence helps to determine fundamental z_s. Here we create broker dealers as intermediaries producing trades. Broker-dealers are agents who try to put together deals, put buyers and sellers of securities together. Formally, the broker-dealer issues (sells) $y(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) \in \mathbb{R}_+$ units of each bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$, at the unit price $P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$. Note that $y(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ at a particular bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ is the number of units of that bundle. There is nothing random. Another distinct bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ has its own quantity, number of units $y(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$. With $\boldsymbol{\theta} \neq \boldsymbol{\theta}'$, the intermediary is taking distinct positions in the market. The clearing constraints below will ensure that when we add up over all bundles, the net positions add up to zero.[25]

Let $\mathbf{y} \in L$ be the vector of the number of bundles issued as one move across the underlying commodity points $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$. With constant returns to scale (see below), the profit of a broker-dealer must be zero and the number of broker-dealers becomes irrelevant. Therefore, without loss of generality, we assume there is one representative broker-dealer, which takes prices as given.

The objective of the broker-dealer is to maximize its profit by supplying $y(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ as follows:

$$\max_{y(\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z},\boldsymbol{\Delta})} \sum_{\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z},\boldsymbol{\Delta}} y(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) \left[P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) - c_{10} - p_0 c_{20} - p_0 k\right] \tag{60}$$

subject to clearing constraints:

$$\sum_{\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z}_{-s},\boldsymbol{\Delta}} y(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}_{-s}, z_s, \boldsymbol{\Delta})\, \theta_{\ell s} = 0,\ \forall s; z_s; \ell = 1, 2, \tag{61}$$

$$\sum_{\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z}_{-s},\boldsymbol{\Delta}} y(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}_{-s}, z_s, \boldsymbol{\Delta})\, \tau_{\ell s} = 0,\ \forall s; z_s; \ell = 1, 2, \tag{62}$$

$$\sum_{\mathbf{c}_0,k,\boldsymbol{\theta},\boldsymbol{\tau},\mathbf{z}_{-s},\boldsymbol{\Delta}} y(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}_{-s}, z_s, \boldsymbol{\Delta})\, \Delta_s = 0\ , \forall s; z_s, \tag{63}$$

again taking prices $p_0, P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ as given. Note that constraints (61) ensure that the books of the broker dealer are balanced in that the value of issuing and holding securities are

[25] Our intermediary is different from Lindahl's producer in that our intermediary does not produce public goods. But the decentralization concept is similar.

equal. Similarly, constraints (62) imply that total supply of collateral including the unwind collateral in a particular exchange is equal to the total commitment to buy the collateral in the same exchange, and constraints (63) ensure that the net supply of the rights to trade in an exchange is zero.

The existence of a maximum to the broker-dealer's problem requires, that for any bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$,

$$P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) \leq c_{10} + p_0 c_{20} + p_0 k + \sum_{\ell,s} \widehat{Q}_\ell(z_s, s)\,\theta_{\ell s} + \sum_{\ell,s} \widehat{P}_\ell(z_s, s)\,\tau_{\ell s} + \sum_s \widehat{P}_\Delta(z_s, s)\,\Delta_s, \tag{64}$$

where $\widehat{Q}_\ell(z_s, s)$, $\widehat{P}_\ell(z_s, s)$ and $\widehat{P}_\Delta(z_s, s)$ are the shadow price of an ex-ante security paying in good ℓ of constraints (61), the shadow price of the spot trade of good ℓ of constraints (62), and the shadow price of the right to trade in the security exchange z_s of constraints (63), respectively. Condition (64) holds with equality if $y(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) > 0$. On the other hand, if the inequality (64) is strict, then $y(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) = 0$ as when implicit shadow costs are greater than revenue.

Market Clearing: The market clearing condition for good 1 in period $t = 0$ is

$$\sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}} y(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})\, c_{10} = \sum_h \alpha^h e_{10}^h \tag{65}$$

Similarly, the market clearing condition for good 2 in period $t = 0$ is

$$\sum_{\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}} y(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})\,[c_{20} + k] = \sum_h \alpha^h e_{20}^h \tag{66}$$

The market clearing conditions for mixtures in period $t = 0$ are

$$\sum_h \alpha^h x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) = y(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}),\ \forall(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}). \tag{67}$$

Definition 6. A competitive equilibrium with segregated exchanges (with mixtures) is a specification of allocation $(\mathbf{x}^h, \mathbf{y})$, and prices $(p_0, P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}))$ such that

(i) for each h, $\mathbf{x}^h \in X^h$ solves utility maximization problem (58) subject to period $t = 0$ budget constraint (59), taking prices as given;

(ii) for the broker-dealer, $\left\{\mathbf{y}, \widehat{Q}_\ell(z_s, s), \widehat{P}_\ell(z_s, s), \widehat{P}_\Delta(z_s, s)\right\}$ solve profit maximization problem (60) subject to clearing-trade constraints (61)-(63) taking prices as given;

(iii) markets for good 1, for good 2, and for mixtures in period $t = 0$ clear, i.e., (65), (66) and (67) hold.

Note that market clearing (67) when substituted into the broker dealer problem gives the Pareto programming problem. Indeed, intuitively, one can go back and forth between the shadow prices of the Pareto problem and the prices and Lagrange multipliers of the consumer and intermediary Lagrangian problems, as first order conditions are necessary and sufficient.

B Proofs

Proof of Proposition 1. We first prove that a competitive collateral equilibrium is constrained optimal if and only if all collateral constraints are not binding, i.e. $\gamma^h_{cc-s} = \mu^h_{cc-s} = 0$ for all h and all s. The proof is based on the first-order conditions for Pareto program (15) and the first-order conditions for a competitive collateral equilibrium. Note that the resource constraints in the Pareto program (15) and the market-clearing constraints in the competitive collateral equilibrium are clearly equivalent. In addition, the collateral constraints are the same in both problems as well. Hence, we only need to match all first-order conditions from both problems. In addition, with limited space, we will focus only on the term that generates an externality.

Optimal Conditions for the Pareto Program (15)

Let μ^h_{cc-s} and $\mu^h_{\bar{u}}$ denote the Lagrange multipliers for the collateral constraint (13) for state s for an agent type h and for the participation constraint (16) for an agent type $h = 1, 2, \ldots, H$ with a normalization of $\mu^1_{\bar{u}} = 1$, respectively. Combining the first-order conditions with respect to c^h_{10} and k^h, and the complementarity slackness conditions for the collateral constraints gives:

$$\begin{aligned}\frac{u^h_{20}}{u^h_{10}} &= \sum_s \pi_s \beta \frac{u^h_{2s}}{u^h_{10}} R_s + \sum_s \frac{\mu^h_{cc-s}}{\mu^h_{\bar{u}} u^h_{10}} p(z_s) R_s + \sum_s \frac{\alpha^h}{\mu^h_{\bar{u}} u^h_{10}} p'(z_s) \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \left[R_s k^h + \theta^{\tilde{h}}_{2s} \right] \\ &= \sum_s \pi_s \beta \frac{u^h_{2s}}{u^h_{10}} R_s + \sum_s \frac{\mu^h_{cc-s}}{\mu^h_{\bar{u}} u^h_{10}} p(z_s) R_s - \sum_s \frac{\alpha^h}{\mu^h_{\bar{u}} u^h_{10}} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu^{\tilde{h}}_{cc-s} \theta^{\tilde{h}}_{1s},\end{aligned} \quad (68)$$

where the last equation follows from the complementarity slackness condition with respect to collateral constraints:

$$\mu_{cc-s}^{\tilde{h}} \left\{ p(z_s) \left[R_s k^{\tilde{h}} + \theta_{2s}^{\tilde{h}} \right] + \theta_{1s}^{\tilde{h}} \right\} = 0 \Rightarrow \mu_{cc-s}^{\tilde{h}} \left[R_s k^{\tilde{h}} + \theta_{2s}^{\tilde{h}} \right] = -\frac{\mu_{cc-s}^{\tilde{h}} \theta_{1s}^{\tilde{h}}}{p(z_s)}. \tag{69}$$

Note that (68) is exactly the same as (17).

Optimal Conditions for a Collateral Equilibrium

Let γ_{cc-s} be the Lagrange multiplier for the collateral constraint for state s. Combining the first-order conditions with respect to c_{10}^h and k^h gives:

$$\frac{u_{20}^h}{u_{10}^h} = \sum_s \pi_s \frac{\beta u_{2s}^h}{u_{10}^h} R_s + \sum_s \frac{\gamma_{cc-s}^h}{u_{10}^h} p(z_s) R_s. \tag{70}$$

We are ready to prove the lemma.

(i) ($\Longleftarrow$) Suppose that $\gamma_{cc-s}^h = \mu_{cc-s}^h = 0$ for all h and all s. We then can show that any competitive collateral equilibrium allocation will also solve the Pareto program (15) by matching all necessary and sufficient conditions. In particular, we can pick $\frac{\mu_{20}}{\mu_{10}} = p_0$, $\frac{\mu_{\ell s}}{\mu_{10}} = Q_{\ell s}$, and $\gamma_{cc-s}^h = \frac{\mu_{cc-s}^h}{\mu_{\bar{u}}^h} = 0$. In conclusion, any collateral equilibrium allocation is constrained optimal if $\gamma_{cc-s}^h = \mu_{cc-s}^h = 0$ for all h and all s.

(ii) ($\Longrightarrow$) Suppose that a competitive collateral equilibrium allocation is constrained optimal, i.e., solves the Pareto program (15). Hence, it must satisfy (68). Using the same matching conditions as above, this will be true only if the last terms in (68) is zero. We will prove this by a contradiction argument.

Suppose that there are some $\tilde{h}$ with $\mu_{cc-s}^{\tilde{h}} \neq 0$, and the last terms in (68) is zero:

$$\frac{\alpha^h}{\mu_{\bar{u}}^h u_{10}^h} \sum_s \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \left(\sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \theta_{1s}^{\tilde{h}} \right) = 0. \tag{71}$$

This must be true for all h and $\tilde{h}$.

We will now argue that $\sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \theta_{1s}^{\tilde{h}}$ has the same *negative* sign for every state s. Using the first-order condition for the Pareto program with respect to θ_{1s}^h, we can show that

$$\sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \theta_{1s}^{\tilde{h}} = \sum_{\tilde{h}} \mu_{1s} \alpha^{\tilde{h}} \theta_{1s}^{\tilde{h}} - \beta \pi_s \sum_{\tilde{h}} \mu_{\bar{u}}^{\tilde{h}} u_{1s}^{\tilde{h}} \theta_{1s}^{\tilde{h}}, \tag{72}$$

where μ_{1s} is the Lagrange multiplier for the resource constraint for θ_{1s}^h. The resource constraint for θ_{1s}^h, $\sum_{\tilde{h}} \alpha^{\tilde{h}} \theta_{1s}^{\tilde{h}} = 0$, then implies that $\sum_{\tilde{h}} \mu_{1s} \alpha^{\tilde{h}} \theta_{1s}^{\tilde{h}} = 0$ for all s. In addition, the first-order condition for the Pareto program with respect to c_{10}^h implies that $\mu_{\bar{u}}^{\tilde{h}} = \frac{\mu_{10}\alpha^{\tilde{h}}}{u_{10}^{\tilde{h}}}$. Thus, we now have

$$\sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \theta_{1s}^{\tilde{h}} = -\beta \pi_s \mu_{10} \sum_{\tilde{h}} \left(\frac{u_{1s}^{\tilde{h}}}{u_{10}^{\tilde{h}}} \right) \alpha^{\tilde{h}} \theta_{1s}^{\tilde{h}}. \tag{73}$$

The optimality requires that an agent with relative large IMRS, $\frac{u_{1s}^{\tilde{h}}}{u_{10}^{\tilde{h}}}$, will hold positive $\theta_{1s}^{\tilde{h}} \geq 0$ and vice versa. This implies that the positive term of $\alpha^{\tilde{h}} \theta_{1s}^{\tilde{h}} \geq 0$ will be weighted more than the negative one. Combining this result with the resource constraint for θ_{1s}^h, $\sum_{\tilde{h}} \alpha^{\tilde{h}} \theta_{1s}^{\tilde{h}} = 0$, we can conclude that $\sum_{\tilde{h}} \left(\frac{u_{1s}^{\tilde{h}}}{u_{10}^{\tilde{h}}} \right) \alpha^{\tilde{h}} \theta_{1s}^{\tilde{h}} \geq 0, \forall s$, and therefore

$$\sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \theta_{1s}^{\tilde{h}} = -\beta \pi_s \mu_{10} \sum_{\tilde{h}} \left(\frac{u_{1s}^{\tilde{h}}}{u_{10}^{\tilde{h}}} \right) \alpha^{\tilde{h}} \theta_{1s}^{\tilde{h}} \leq 0, \forall s. \tag{74}$$

With strictly concave and identical homothetic utility function, we can show that $\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} < 0$, and therefore can conclude that

$$\frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \left(\sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \theta_{1s}^{\tilde{h}} \right) \geq 0, \ \forall s. \tag{75}$$

As a result, (71) will hold only if

$$\sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \theta_{1s}^{\tilde{h}} = -\beta \pi_s \mu_{10} \sum_{\tilde{h}} \left(\frac{u_{1s}^{\tilde{h}}}{u_{10}^{\tilde{h}}} \right) \alpha^{\tilde{h}} \theta_{1s}^{\tilde{h}} = 0, \forall s. \tag{76}$$

Given that $\sum_{\tilde{h}} \alpha^{\tilde{h}} \theta_{1s}^{\tilde{h}} = 0$, condition (76) implies that $\frac{u_{1s}^{\tilde{h}}}{u_{10}^{\tilde{h}}} = \frac{u_{1s}^h}{u_{10}^h}$, $\forall h, \tilde{h}; s$. Using the fact that $\frac{u_{2s}^h}{u_{1s}^h} = p(z_s)$ for all h, we can also show that $\frac{u_{1s}^{\tilde{h}}}{u_{1\hat{s}}^{\tilde{h}}} = \frac{u_{1s}^h}{u_{1\hat{s}}^h}$, $\forall h, \tilde{h}; s$. In words, the marginal rate of substitutions across times and states are equalized across agent types. Under the assumption stated in the proposition, these equalities are necessary and sufficient conditions for first-best optimality, which in turn implies that all collateral constraints are not binding, i.e., $\gamma_{cc-s}^h = \mu_{cc-s}^h = 0$ for all h and all s. Hence, we can conclude that a collateral equilibrium is constrained optimal, solving the Pareto program (15), only if all collateral constraints are not binding.

The rest of the proof is by contrapositive. Suppose a competitive collateral equilibrium is constrained optimal. The above result implies that a necessary and sufficient condition for a competitive collateral equilibrium to be constrained optimal is that all collateral constraints are not binding. No binding collateral constraints implies first-best optimality. In short, we have shown that first-best optimality is a necessary and sufficient condition for constrained optimality. Thus we can conclude that a competitive collateral equilibrium is constrained *suboptimal* if and only if it is not first-best optimal. □

Proof of Proposition 2. The proof is an immediate result of the proof of proposition 1 above. First, if a competitive collateral equilibrium is not first-best optimal, then (by Proposition 1) we can show that the last term of (68) is strictly positive:

$$\sum_s \frac{\alpha^h}{\mu_{\bar{u}}^h u_{10}^h} \frac{p'(z_s)}{p(z_s)} \frac{\partial z_s}{\partial K} \sum_{\tilde{h}} \mu_{cc-s}^{\tilde{h}} \theta_{1s}^{\tilde{h}} > 0. \tag{77}$$

This implies that the marginal rate of substitution between good 1 and good 2 in period $t = 0$ at the competitive collateral equilibrium is larger than the optimal level of the marginal rate of substitution between good 1 and good 2 in period $t = 0$, i.e., $\left.\frac{u_{20}^h}{u_{10}^h}\right|_{ce} > \left.\frac{u_{20}^h}{u_{10}^h}\right|_{op}$. This implies that the equilibrium price of good 2 in period $t = 0$ is too high relative to its shadow price from the (constrained) optimal allocation $\left.\frac{u_{20}^h}{u_{10}^h}\right|_{op}$. In addition, given that the aggregate consumption of good 1 is fixed and preferences are identically homothetic, this result can be true only if the (endogenous) aggregate saving/collateral in a competitive collateral equilibrium, K^{ce}, is too large, i.e., $K^{ce} > K^{op}$. □

Proof of Theorem 1. Let $(\mathbf{x}, \mathbf{y})$, and $(p_0, P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}))$ be a competitive equilibrium. Suppose the competitive equilibrium allocation is not Pareto optimal, i.e. there is an attainable allocation $\tilde{\mathbf{x}} \in X$ such that $U^h(\tilde{\mathbf{x}}^h) \geq U^h(\mathbf{x}^h)$ for all h and $U^h\left(\tilde{\mathbf{x}}^{\tilde{h}}\right) > U^h\left(\mathbf{x}^{\tilde{h}}\right)$ for some $\tilde{h}$. For notational purposes, let $b \equiv (\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ be a typical bundle. With local nonsatiation of preferences, we have $\sum_b P(b) x^h(b) \leq \sum_b P(b) \tilde{x}^h(b)$ for all h, and $\sum_b P(b) x^{\hat{h}}(b) < \sum_b P(b) \tilde{x}^{\hat{h}}(b)$ for some $\hat{h}$. Summing over all agents with weights α^h, we have

$$\sum_b P(b) \sum_h \alpha^h x^h(b) < \sum_b P(b) \sum_h \alpha^h \tilde{x}^h(b). \tag{78}$$

The optimal condition (64) for the market-maker's profit maximization problem implies that, for any typical bundle b,

$$P(b)y(b) = \left(c_{10} + p_0 c_{20} + p_0 k + \sum_{s,\ell} \widehat{Q}_\ell (z_s, s)\, \theta_{\ell s} + \sum_{s,\ell} \widehat{P}_\ell (z_s, s)\, \tau_{\ell s} + \sum_s \widehat{P}_\Delta (z_s, s)\, \Delta_s \right) y(b). \tag{79}$$

Using the market-clearing condition for mixtures in period $t = 0$, (67), we can substitute $\sum_h \alpha^h x^h(b)$ for $y(b)$ for every bundle b on the left hand side. Then, summing over all bundles b gives

$$\sum_b P(b) \sum_h \alpha^h x^h(b) = \sum_h \alpha^h e_{10}^h + p_0 \sum_h \alpha^h e_{20}^h, \tag{80}$$

where we apply the technology constraints of the broker-dealer (61)-(63) and the market clearing conditions (65)-(66). Similarly, from what we know already about expenditures for the supposed dominating bundle at the outset of this proof, we can also show that

$$\sum_b P(b) \sum_h \alpha^h \tilde{x}^h(b) \leq \sum_h \alpha^h e_{10}^h + p_0 \sum_h \alpha^h e_{20}^h. \tag{81}$$

Using (80) and (81), (78) can be rewritten as

$$\sum_h \alpha^h e_{10}^h + p_0 \sum_h \alpha^h e_{20}^h < \sum_h \alpha^h e_{10}^h + p_0 \sum_h \alpha^h e_{20}^h$$

This is a contradiction! □

Proof of Theorem 2. We will first prove that any constrained optimal allocation can be decentralized as a compensated equilibrium. Then, we will use a standard cheaper-point argument (see Debreu, 1954) to show that any compensated equilibrium is a competitive equilibrium with transfers. The compensated equilibrium is defined as follows.

Definition 7. A compensated equilibrium is a specification of allocation $(\mathbf{x}, \mathbf{y})$, and prices p_0, $P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ such that

(i) for each h as a price taker, $\mathbf{x}^h \in X^h$ solves

$$\min_{\hat{\mathbf{x}}^h} \sum_{(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})} P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})\, \hat{x}^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) \tag{82}$$

subject to

$$U^h\left(\hat{\mathbf{x}}^h\right) \geq U^h\left(\mathbf{x}^h\right); \tag{83}$$

(ii) for the market-maker, $\left\{\mathbf{y}, \widehat{Q}_\ell(z_s, s), \widehat{P}_\ell(z_s, s), \widehat{P}_\Delta(z_s, s)\right\}$ solves (60), taking prices as given,

(iii) in period-0, markets for good-1, good-2 and mixtures clear, i.e., (65)-(67) hold.

Given that the optimization problems are well-defined concave problems, Kuhn-Tucker conditions are necessary and sufficient. The proof are divided into three steps.

(i) Kuhn-Tucker conditions for a compensated equilibrium allocation: Let $\gamma^h(0)$ and $\gamma^h(l)$ be the Lagrange multiplier for constraint (83), and for the probability constraint (47), respectively. The optimal condition for $x^h(c, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z})$ is given by

$$\gamma^h(0)V^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) \leq P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) + \gamma^h(l), \tag{84}$$

where the inequality holds with equality if $x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) > 0$. Recall that the optimal condition (64) for the market-maker's profit maximization problem implies that, for any typical bundle b,

$$P(b) \leq \left(c_{10} + p_0 c_{20} + p_0 k + \sum_{s,\ell} \widehat{Q}_\ell(z_s, s)\,\theta_{\ell s} + \sum_{s,\ell} \widehat{P}_\ell(z_s, s)\,\tau_{\ell s} + \sum_s \widehat{P}_\Delta(z_s, s)\,\Delta_s\right). \tag{85}$$

where the condition holds with equality if $y(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) > 0$.

(ii) Kuhn-Tucker conditions for Pareto optimal allocations: A solution to the Pareto program satisfies condition (56). In addition, for any bundle $(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ with $x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) > 0$ for some h, we can show that $\frac{\widetilde{Q}_2(z_s,s)}{\widetilde{Q}_1(z_s,s)} = \frac{u^h_{2s}}{u^h_{1s}} = p(z_s)$. This result is derived using a variational principle with respect to θ_{1s} and θ_{2s}, and using the fact that the agent can trade in spot markets at price $p(z_s)$, which implies that $\frac{u^h_{2s}}{u^h_{1s}} = p(z_s)$.

(iii) Matching dual variables and prices: We then set $\gamma^h(0) = \frac{\lambda^h}{\widetilde{P}_{10}}$, $p_0 = \frac{\widetilde{P}_{20}}{\widetilde{P}_{10}}$, $\widehat{Q}_\ell(z_s, s) = \frac{\widetilde{Q}_\ell(z_s,s)}{\widetilde{P}_{10}}$, $\widehat{P}_\ell(z_s, s) = \frac{\widetilde{P}_\ell(z_s,s)}{\widetilde{P}_{10}}$, $\widehat{P}_\Delta(z_s, s) = \frac{\widetilde{P}_\Delta(z_s,s)}{\widetilde{P}_{10}}$ and $\gamma^h(l) = \frac{\widetilde{P}^h_l}{\widetilde{P}_{10}}$, which imply that the optimal conditions of the Pareto program are equivalent to the optimal conditions for consumers' and market-maker's problems in the compensated equilibrium. That is, a solution to the Pareto program also solves the consumer's and market-maker's problems. Again the resource and consistency constraints in the Pareto program are identical to the market-clearing and consistency conditions in equilibrium. To sum up, any Pareto optimal allocation is a compensated equilibrium.

Next we show that any compensated equilibrium, corresponding to $\lambda^h > 0$, is a competitive equilibrium with transfers using the cheaper point argument. First of all, let the wealth of an agent type h in the compensated equilibrium be $w^h = \sum_b P(b)\, x^h(b)$, which is feasible, i.e. $\sum_h \alpha^h w^h = \sum_h \alpha^h \left(e_{10}^h + p_0 e_{20}^h\right)$. In addition, with $\lambda^h > 0$, for every h, an Inada condition guarantees that a solution to the Pareto program, which is a compensated equilibrium allocation, will not have a strictly positive mass on $c = 0$.

We can pick a cheaper allocation as $\hat{\mathbf{x}}$, with $\mathbf{c}_0 = 0$. More specifically, let $0 \in C$, and set $\hat{x}^h(0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) = \sum_{\mathbf{c}_0} x^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$ and $\hat{x}^h(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) = 0$, for any $\mathbf{c}_0 \neq 0$. Note that the alternative allocation put strictly positive masses on bundles with $\mathbf{c}_0 = 0$. The strictly increasing utility function implies that $p_0 > 0$. Consequently, the optimal condition of the market-maker (64) implies that $P(\mathbf{c}_0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta}) > P(0, k, \boldsymbol{\theta}, \boldsymbol{\tau}, \mathbf{z}, \boldsymbol{\Delta})$, for any $\mathbf{c}_0 \geq \mathbf{0}$ and $\mathbf{c}_0 \neq \mathbf{0}$. As a result, we can show that $\sum_b P(b)\, x^h(b) > \sum_b P(b)\, \hat{x}^h(b)$.

To sum up, we have shown that there exists an allocation $\hat{\mathbf{x}}^h$ that is cheaper than the compensated equilibrium allocation, $\mathbf{x}^h$, for every agent h. As a result, using the cheaper-point argument, a compensated equilibrium is a competitive equilibrium with transfers. $\square$

Proof of Theorem 3. For notational convenience, we redefine the grid to include the endowment profiles, i.e.,

$$
\begin{aligned}
\mathbf{e}^h(b) &= 1, \quad \text{for } b = (\mathbf{e}_0, k = 0, \theta = 0, \tau = 0, \mathbf{z} = 0, \Delta = 0) \\
&= 0, \quad \text{otherwise}
\end{aligned}
$$

In addition, the optimal condition of the market-maker (64) implies that the price of bundle $\left(\mathbf{e}_0^h, k = 0, \theta = 0, \tau = 0, \mathbf{z} = 0, \Delta = 0\right)$ is $P\left(\mathbf{e}_0^h, 0, 0, 0, 0, 0\right) = e_{10}^h + p_0 e_{20}^h$. Therefore, the total value of period-0 endowment lottery of an agent h, $\mathbf{e}^h$, is given by

$$
\sum_b P(b)\, \mathbf{e}^h(b) = P\left(\mathbf{e}_0^h, 0, 0, 0, 0, 0\right) = e_{10}^h + p_0 e_{20}^h \tag{86}
$$

which is exactly income in the budget constraint (59).

Let $\mathbf{P} = [P(b)]_b$ be the prices of all bundles. In addition, we also add the price of good 2 in period $t = 0$, p_0 into the price space as $p_0 = P(c = (0, 1), 0, 0, 0, 0, 0)$. As in Prescott and

Townsend (2005), with the possibility of negative prices, we restrict prices $\mathbf{P}$ to the closed unit ball;

$$D = \left\{ \mathbf{P} \in \mathbb{R}^n | \sqrt{\mathbf{P} \cdot \mathbf{P}} \leq 1 \right\}, \tag{87}$$

where "·" is the inner product operator. Note that the set D is compact and convex.

Consider the following mapping $(\lambda, \mathbf{x}, \mathbf{P}) \to (\lambda', \mathbf{x}', \mathbf{P}')$, where $\lambda, \lambda' \in S^{H-1}$, $\mathbf{x}^h \in X^h$. Recall that the consumption possibility set X^h is non-empty, convex, and compact. Let $\overline{X}$ be the cross-product over h of X^h: $\overline{X} = X^1 \times \ldots \times X^H$.

The first part of the mapping is given by $\lambda \longrightarrow (\mathbf{x}', \mathbf{P}')$, where $\mathbf{x}'$ is the solution to the Pareto program given the Pareto weight λ, and $\mathbf{P}'$ is the renormalized prices. With the second welfare theorem, the solution to the Pareto program for a given Pareto weight λ also gives us (compensated) equilibrium prices $\mathbf{P}^*$. The nonlocal satiation of preferences implies that $\mathbf{P}^* \neq 0$. The normalized prices are given by

$$\mathbf{P}' = \frac{\mathbf{P}^*}{\mathbf{P}^* \cdot \mathbf{P}^*}$$

Note that $\mathbf{P}' \cdot \mathbf{P}' = 1$. In order to preserve the convexity of the mapping while prices in the unit ball D, we define the convex hull of the normalized prices. Let $\widetilde{D}$ be the sets of all normalized prices, and accordingly $co\widetilde{D}$ be its convex hull. Since $\mathbf{P}' \in \widetilde{D}$, $\mathbf{P}' \in co\widetilde{D}$, which is compact and convex. Note that extending $\widetilde{D}$ to its convex hull does not add any new relative prices. It is not too difficult to show that this mapping, $\lambda \longrightarrow (\mathbf{x}', \mathbf{P}')$, is non-empty, compact-valued, convex-valued. By the Maximum theorem, it is upper hemi-continuous. In addition, the upper hemi-continuity is preserved under the convex-hull operation.

The second part of the mapping is given by $(\lambda, \mathbf{x}, \mathbf{P}) \longrightarrow \lambda'$. The new weight can be formed as follows:

$$\hat{\lambda}^h = \max \left\{ 0, \lambda^h + \frac{\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h \right)}{A} \right\} \tag{88}$$

$$\lambda'^h = \frac{\hat{\lambda}^h}{\sum_h \hat{\lambda}^h} \tag{89}$$

where A is a positive number such that $\sum_h \left| \mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h \right) \right| \leq A$. It is clear that this mapping is also non-empty, compact-valued, convex-valued, and upper hemi-continuous. In conclusion, $(\lambda, \mathbf{x}, \mathbf{P}) \to (\lambda', \mathbf{x}', \mathbf{P}')$ is a mapping from $S^{H-1} \times \overline{X} \times S^{n-1} \longrightarrow S^{H-1} \times \overline{X} \times S^{n+1}$.

Since each set is non-empty, compact, and convex, so does its cross-product. In addition, the overall mapping is non-empty, compact-valued, convex-valued, and upper hemi-continuous since these properties are preserved under the cross product operation. By Kakutani's fixed point theorem, there exists a fixed point $(\lambda, \mathbf{x}, \mathbf{P})$.

Proved in Theorem 2, any Pareto optimal allocation can be supported as a compensated equilibrium. In addition, the strictly increasing utility function implies that $p_0 > 0$. Hence, with positive endowments, an agent h's wealth at the fixed point is strictly positive;

$$w^h = \mathbf{P} \cdot \mathbf{e}^h = e_{10}^h + p_0 e_{20}^h > 0$$

With strictly positive wealth, a compensated equilibrium is a competitive equilibrium with transfers (using a cheaper-point argument as in the proof of Theorem 2).

We now need to show that there is no need for wealth transfers in equilibrium, i.e., the budget constraint without transfers

$$\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0$$

holds for every agent h. It is not difficult to show that

$$\sum_h \alpha^h \mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) \;\; = \;\; 0$$

In addition, at a fixed point $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right)$ must be the same sign for every h. Hence, $\mathbf{P} \cdot \left(\mathbf{e}^h - \mathbf{x}^h\right) = 0$ for every agent h. This clearly confirms that the budget constraint (without transfers) of every agent h holds. Hence, a competitive equilibrium (without transfers) exists. $\square$

C More Results

Lemma 1. *For any state-contingent security, there exists a security with* no default *that can generate the* same total payoffs *using the* same amount of collateral*.*

Proof of Lemma 1: Default is Irrelevant under Complete Contracts. Consider a contingent security *that will be in default* in state s, with collateral $\widehat{C} < \frac{1}{R_s p(z_s)}$. That is, an issuer of this security will "default" in state s. Hence, according to condition (**??**), the payoff of this security (in units of good 1) in state s is

$$\min\left(1, \widehat{C}R_s p(z_s)\right) = \widehat{C}R_s p(z_s) < 1. \tag{90}$$

We now argue that there is an alternative security that does not default but generates exactly the same total payoffs using the same amount of collateral overall. Consider a state-s contingent security with collateral amount $\widetilde{C} = \frac{1}{R_s p(z_s)}$. This security will not default. It is straightforward to show that the payoff of this security is one unit of good 1 in state s. Now consider $\widehat{C}R_s p(z_s)$ units of the *alternative* security. That collection of securities pays in state s one per unit or $\widehat{C}R_s p(z_s)$ in total. This is exactly the same as the payoff of the original security with default: see (90). In addition, the total collateral for $\widehat{C}R_s p(z_s)$ units of the alternative security with $\frac{1}{R_s p(z_s)}$ collateral per unit is $\widehat{C}$, which is exactly the same as the collateral level of the original security. Therefore, the alternative security can generate the same payoffs using the same total amount of collateral but without default. A similar argument also applies to all other types of securities. $\square$

C.1 Details of the Building Blocks of the Collateral Constraints

This section precisely defines directly collateralized and asset-back securities (pyramiding), and derives the unified collateral constraints (1) by considering the collateral constraints of each type of securities one at a time and adding them up (and disaggregating back down).

Collateral Constraints on Directly Collateralized Securities

Let ψ_{1s}^h and ψ_{2s}^h denote agent h's demand at the end of period 0 for a security paying in good 1 and in good 2, both with good 2 as collateral directly, respectively. Again, we adopt the convention that positive means demand and negative means sale. So, holding a positive amount of a security paying good 2 in state s, $max\left(0, \psi_{2s}^h\right) = \psi_{2s}^h$, a positive number, is equivalent to buying that security (or lending) while holding a negative amount of a security, $min\left(0, \psi_{2s}^h\right) = \psi_{2s}^h$, a negative number, is equivalent to selling that security (or borrowing). In short, the max and min operators pick off demand and supply, respectively. A wedge is created by the need to back the supply by collateral but not the demand.

More generally, a security paying a unit of good 1 in state s backed by good 2 pays the minimum of 1 unit of good 1 or the value of its collateral in state s. By an argument similar to the one given earlier, the minimum no-default collateral is $\frac{1}{p(z_s)R_s}$ per unit. Similarly, with no-default and no-over-collateralization, a security paying in good 2 in state s requires $\frac{1}{R_s}$ units of good 2 as collateral. The results so far are summarized in the first two rows of the Table 6 with collateral requirement in the last column.

Table 6: Collateral requirements for each type of securities.

	payment unit	collateral unit	issued liabilities	purchased assets available as collateral	total collateral requirement for no default securities
ψ^h_{1s}	good 1	good 2	$-\min\left(0, \psi^h_{1s}\right)$	$\max\left(0, \psi^h_{1s}\right)$	$-\left(\frac{1}{R_s p(z_s)}\right)\min\left(0, \psi^h_{1s}\right)$
ψ^h_{2s}	good 2	good 2	$-\min\left(0, \psi^h_{2s}\right)$	$\max\left(0, \psi^h_{2s}\right)$	$-\left(\frac{1}{R_s}\right)\min\left(0, \psi^h_{2s}\right)$
σ^h_{1s}	good 1	securities paying in good 2	$-\min\left(0, \sigma^h_{1s}\right)$	$\max\left(0, \sigma^h_{1s}\right)$	$-\left(\frac{1}{p(z_s)}\right)\min\left(0, \sigma^h_{1s}\right)$
σ^h_{2s}	good 2	securities paying in good 1	$-\min\left(0, \sigma^h_{2s}\right)$	$\max\left(0, \sigma^h_{2s}\right)$	$-p(z_s)\min\left(0, \sigma^h_{2s}\right)$
ν^h_{1s}	good 1	securities paying in good 1	$-\min\left(0, \nu^h_{1s}\right)$	$\max\left(0, \nu^h_{1s}\right)$	$-\min\left(0, \nu^h_{1s}\right)$
ν^h_{2s}	good 2	securities paying in good 2	$-\min\left(0, \nu^h_{2s}\right)$	$\max\left(0, \nu^h_{2s}\right)$	$-\min\left(0, \nu^h_{2s}\right)$

For securities $\left(\psi^h_{1s}, \psi^h_{2s}\right)$ with good 2 as collateral, paying in good 1 and good 2, respectively, agent h must hold good 2 at the end of period 0 no less than the collateral requirement in any state (shown in Table 6):

$$k^h \geq -\min\left(0, \psi^h_{1s}\right)\left(\frac{1}{R_s p(z_s)}\right) - \min\left(0, \psi^h_{2s}\right)\left(\frac{1}{R_s}\right), \ \forall s, \tag{91}$$

which can be rewritten as

$$p(z_s)R_s k^h + \min\left(0, \psi^h_{1s}\right) + p(z_s)\min\left(0, \psi^h_{2s}\right) \geq 0, \ \forall s. \tag{92}$$

These are *state-contingent collateral requirement constraints* with directly collateralized securities. We incorporate asset-backed securities in the next section.

Note that when an agent h's collateral requirement constraints (91) are not binding for every state s (i.e., the LHS of (91) exceeds its RHS or (91) holds with strict inequality for every state s), then the agent h holds collateral k^h more than needed to back issued securities. The extra part of collateral is normal saving.

Pyramiding: Asset-Backed Securities

In real world economies, agents are allowed to use the *promises to receive* goods of others as collateral to back their own promises. This is termed *pyramiding.* In other words, there are two types of collateral, good 2 itself (described in the preceding section) and "assets" backed by such collateral. The prototypical example of an asset-backed promise in this paper is an ex-ante agreement for an agent to give up good 1 in the spot market in state s backed by someone else's promise, a receipt of good 2, or vice versa. The promise of receipt is the asset, and this backs the promise to pay. Indeed, if the planned spot-market trade is at equilibrium price of $p(z_s)$, then one is moving along a budget line and so the value of collateral, the good to be recovered, exactly equals the promise and there is no need for additional underlying collateral.

With two physical commodities, there are four possible types of asset-backed securities, summarized in the last four rows of Table 6. For example, a unit of an asset-backed security $\hat{\sigma}_s$ paying in good 1 in state s needs $\frac{1}{p(z_s)}$ units of assets paying in good 2 as collateral. The value of the payoff of $\frac{1}{p(z_s)}$ units of securities paying in good 2 in state s equals $p(z_s)\times\frac{1}{p(z_s)} = 1$ unit of good 1, which is exactly the face-value promise to pay. These collateral requirements are minimum no-default levels.

As shown in the third row of Table 6 (see the column titled total collateral requirement), an asset-backed security paying a unit of good 1 in state s, σ_{1s}^h, requires that the total amount of purchased assets paying in good 2 in state s is no less than $-\left(\frac{1}{p(z_s)}\right)\min\left(0,\sigma_{1s}^h\right)$. Similarly, an asset-backed security ν_{2s}^h requires that the total amount of purchased assets paying in good 2 in state s is no less than $-\min\left(0,\nu_{2s}^h\right)$ (see the last row of Table 6). On the other hand, the total amount of purchased assets paying in good 2 is $\max\left(0,\psi_{2s}^h\right)+\max\left(0,\sigma_{2s}^h\right)+\max\left(0,\nu_{2s}^h\right)$, as shown in the second, fourth and last rows of Table 6 (see the next-to-last column titled purchased assets). Hence, the collateral requirement condition regarding issued securities

σ_{1s}^h and ν_{2s}^h that require financial assets paying in good 2 as collateral can be written as, for any state s,

$$\max\left(0,\psi_{2s}^h\right)+\max\left(0,\sigma_{2s}^h\right)+\max\left(0,\nu_{2s}^h\right) \quad\geq\quad -\left(\frac{1}{p(z_s)}\right)\min\left(0,\sigma_{1s}^h\right)-\min\left(0,\nu_s^h\right).$$

This states that the agent purchases enough assets or promises paying in good 2, $\theta_{2s}^h,\sigma_{2s}^h,\nu_{2s}^h$, to back up her own asset-backed securities or issued promises σ_{1s}^h,ν_{2s}^h. The above condition can be rearranged as

$$p(z_s)\max\left(0,\psi_{2s}^h\right)+p(z_s)\max\left(0,\sigma_{2s}^h\right)+p(z_s)\nu_{2s}^h \quad\geq\quad -\min\left(0,\sigma_{1s}^h\right), \tag{93}$$

where we applies the fact that $\max\left(0,\nu_{2s}^h\right)+\min\left(0,\nu_{2s}^h\right)=\nu_{2s}^h$.

Similarly, the collateral requirement condition for issued securities that require financial assets paying in good 1 as collateral is given by

$$\max\left(0,\psi_{1s}^h\right)+\max\left(0,\sigma_{1s}^h\right)+\nu_{1s}^h \quad\geq\quad -p(z_s)\min\left(0,\sigma_{2s}^h\right),\ \forall s, \tag{94}$$

where the right-hand-side comes from the fourth and fifth rows of Table 6.

We now show that the collateral constraints

$$p(z_s)R_sk^h+\theta_{1s}^h+p(z_s)\theta_{2s}^h \quad\geq\quad 0,\ \forall s \tag{95}$$

are equivalent to collateral requirement conditions (with three types of collateral), (92), (93), and (94). In other words, there is no loss of generality to use the collateral constraints (95); an allocation is attainable under (95) if and only if it is so under (92), (93), and (94).

To be more precise, let $\theta_{1s}^h=\psi_{1s}^h+\sigma_{1s}^h+\nu_{1s}^h$ and $\theta_{2s}^h=\psi_{2s}^h+\sigma_{2s}^h+\nu_{2s}^h$ be contingent securities paying in good 1 and in good 2 in state s, respectively, which can be backed either by good 2 or purchased assets (other people's promises). Note that θ_{1s}^h and θ_{2s}^h include both directly collateralized and asset-backed securities. An attainable allocation under (92), (93), and (94) can be defined similarly to the one under (1) by replacing (7) the following resource constraints:

$$\sum_h\alpha^h\psi_{1s}^h=\sum_h\alpha^h\psi_{2s}^h=\sum_h\alpha^h\sigma_{1s}^h=\sum_h\alpha^h\sigma_{2s}^h=\sum_h\alpha^h\nu_{1s}^h=\sum_h\alpha^h\nu_{2s}^h=0,\ \forall s. \tag{96}$$

The collateral constraint (95) results from summing (92), (93), and (94) altogether, and then applying $\max(0,x)+\min(0,x)=x$ to get rid of max and min operators. In addition, the proof of this lemma also shows how to recover contract allocation $\left(\psi^h_{1s}, \psi^h_{2s}, \sigma^h_{1s}, \sigma^h_{2s}, \nu^h_{1s}, \nu^h_{2s}\right)_h$ from $\left(\theta^h_{1s}, \theta^h_{2s}\right)$.

Lemma 2. *The following statements are true:*

(i) if $\left(\mathbf{c}^h_0, k^h, \psi^h_{1s}, \psi^h_{2s}, \sigma^h_{1s}, \sigma^h_{2s}, \nu^h_{1s}, \nu^h_{2s}\right)_h$ is attainable, then the collateral constraint (95) and the market-clearing conditions (7) hold, and

(ii) if $\left(k^h, \theta^h_{1s}, \theta^h_{2s}\right)_h$ is attainable, then there exists a collateral and security allocation $\left(k^h, \psi^h_{1s}, \psi^h_{2s}, \sigma^h_{1s}, \sigma^h_{2s}, \nu^h_{1s}, \nu^h_{2s}\right)_h$ that satisfies collateral requirement conditions (92), (93), (94) and the market-clearing conditions (96).

Proof. The first statement can be proved as follows. First, it is clear that conditions (96) imply (7). We now only need to show that (92), (93), and (94) imply (95). Summing up all collateral requirement conditions, (92), (93), and (94), and using the fact that $\max(0, x) + \min(0, x) = x$ give, for an agent h in state s,

$$p(z_s)R_s k^h + \left[\psi^h_{1s} + \sigma^h_{1s} + \nu^h_{1s}\right] + p(z_s)\left[\psi^h_{2s} + \sigma^h_{2s} + \nu^h_{2s}\right] \quad \geq \quad 0,$$

which is the collateral constraint for an agent h in state s where $\theta^h_{1s} = \psi^h_{1s} + \sigma^h_{1s} + \nu^h_{1s}$ and $\theta^h_{2s} = \psi^h_{2s} + \sigma^h_{2s} + \nu^h_{2s}$.

The second statement is proved as follows. Consider an allocation $\left(k^h, \theta^h_{1s}, \theta^h_{2s}\right)_h$ that satisfies (95) and (7). We will now choose a corresponding allocation $\left(k^h, \psi^h_{1s}, \psi^h_{2s}, \sigma^h_{1s}, \sigma^h_{2s}, \nu^h_{1s}, \nu^h_{2s}\right)_h$ that satisfies $\theta^h_{1s} = \psi^h_{1s} + \sigma^h_{1s} + \nu^h_{1s}$, $\theta^h_{2s} = \psi^h_{2s} + \sigma^h_{2s} + \nu^h_{2s}$, the collateral requirement conditions (92), (93), (94), and the market-clearing conditions (96). Consider the following candidate allocation:

$$\psi^h_{1s} = \theta^h_{1s} + p(z_s)\theta^h_{2s}, \tag{97}$$

$$\psi^h_{2s} = \nu^h_{1s} = \nu^h_{2s} = 0, \tag{98}$$

$$\sigma^h_{1s} = \theta^h_{1s} - \psi^h_{1s} = -p(z_s)\theta^h_{2s}, \tag{99}$$

$$\sigma^h_{2s} = \theta^h_{2s}. \tag{100}$$

(98) implies that agents hold no $\psi^h_{2s}, \nu^h_{1s}, \nu^h_{2s}$; they will borrow or lend through directly collateralized contract paying in good 1 ψ^h_{1s} only.

It is straightforward to show that resource constraints (96) hold. Since the resource constraints are satisfied and the collateral allocations k^h are the same, the market fundamentals are the same. We now would like to show that collateral requirement conditions (92), (93), (94) also hold. First, we will show that (93) and (94) hold. There are two cases to consider; (i) $\theta_{2s}^h > 0$, (ii) $\theta_{2s}^h < 0$. Case I: Suppose that $\theta_{2s}^h > 0$. Using (100), this implies that $\sigma_{2s}^h > 0$, which in turn leads to $\min\left(0, \sigma_{2s}^h\right) = 0$. On the other hand, it is true that

$$\max\left(0, \psi_{1s}^h\right) + \max\left(0, \sigma_{1s}^h\right) = \max\left(0, \psi_{1s}^h\right) + \max\left(0, \sigma_{1s}^h\right) + \nu_{1s}^h \geq 0,$$

where the first equality follows from (98). Since $\min\left(0, \sigma_{2s}^h\right) = 0$, we have

$$\max\left(0, \psi_{1s}^h\right) + \max\left(0, \sigma_{1s}^h\right) = \max\left(0, \psi_{1s}^h\right) + \max\left(0, \sigma_{1s}^h\right) + \nu_{1s}^h \geq -p(z_s)\min\left(0, \sigma_{2s}^h\right),$$

which is (94). On the other hand, (99) implies that $\sigma_{1s}^h < 0$ when $\theta_{2s}^h > 0$. As a result, $\min\left(0, \sigma_{1s}^h\right) = \sigma_{1s}^h$. Using (98), (99), (100), we then can show that

$$\begin{aligned} &p(z_s)\max\left(0, \psi_{2s}^h\right) + p(z_s)\max\left(0, \sigma_{2s}^h\right) + p(z_s)\nu_{2s}^h + \min\left(0, \sigma_{1s}^h\right) \\ &\qquad = 0 + p(z_s)\sigma_{2s}^h + 0 + \sigma_{1s}^h = p(z_s)\theta_{2s}^h - p(z_s)\theta_{2s}^h = 0, \end{aligned}$$

where the second equality follows from (99) and (100). This shows that (93) holds.

Case II: Suppose that $\theta_{2s}^h < 0$. (99) and (100) imply that $\max\left(0, \sigma_{1s}^h\right) = \sigma_{1s}^h = -p(z_s)\theta_{2s}^h$ and $\min\left(0, \sigma_{2s}^h\right) = \sigma_{2s}^h = \theta_{2s}^h$, respectively. We then can write

$$\max\left(0, \psi_{1s}^h\right) + \max\left(0, \sigma_{1s}^h\right) + \nu_{1s}^h = \max\left(0, \psi_{1s}^h\right) - p(z_s)\theta_{2s}^h \geq -p(z_s)\theta_{2s}^h = -p(z_s)\min\left(0, \sigma_{2s}^h\right),$$

which is exactly (94). Note that the first equality follows from (98), the second inequality follows from the fact that $\max\left(0, \psi_{1s}^h\right) \geq 0$. Similarly, using , we can show that $\max\left(0, \sigma_{2s}^h\right) = \min\left(0, \sigma_{1s}^h\right) = 0$. This implies that

$$p(z_s)\max\left(0, \psi_{2s}^h\right) + p(z_s)\max\left(0, \sigma_{2s}^h\right) + p(z_s)\nu_{2s}^h + \min\left(0, \sigma_{1s}^h\right) = 0 + 0 + 0 + 0 = 0,$$

which is exactly (93).

Similarly, we can now show that (92) also holds. There are two cases to be considered as well.

Case I: suppose that $\theta_{1s}^h + p(z_s)\theta_{2s}^h < 0$. (97) implies that $\psi_{1s}^h < 0$, which in turn implies that $\min\left(0, \psi_{1s}^h\right) = \psi_{1s}^h = \theta_{1s}^h + p(z_s)\theta_{2s}^h$. Using (98), we now can show that

$$p(z_s)R_s k^h + \min\left(0, \psi_{1s}^h\right) + p(z_s)\min\left(0, \psi_{2s}^h\right) = p(z_s)R_s k^h + \theta_{1s}^h + p(z_s)\theta_{2s}^h + 0 \geq 0,$$

where the last inequality follows (95). This implies that (92) holds.

Case II: we can use a similar argument to show that (92) holds when $\theta_{1s}^h + p(z_s)\theta_{2s}^h = \psi_{1s}^h > 0$. In summary, we have show that all collateral requirement conditions hold. □

C.2 Ex-ante Contracting versus Ex-post Spot Trading

Thus far we implicitly shut down trade in the spot markets in each state. This section shows that the spot markets are redundant when all types of contracts are available (see Lemma 3 below). In other words, agents do not need to trade in spot markets, though they may well do so. Importantly, the spot markets are open and deliver the spot price $p(z_s)$. In addition, we also show that the asset-backed securities are not necessary when the spot markets are open and active (see Lemma 4 below). Put differently, agents simply are indifferent between trading in spot markets or ex-ante asset-backed securities.

When the spot markets are open, each agent h can trade τ_{1s}^h units of good 1 for τ_{2s}^h units of good 2 at a spot price $p(z_s)$ according to the spot-trade constraint:

$$\tau_{1s}^h + p(z_s)\tau_{2s}^h = 0. \tag{101}$$

Recall that the spot price function, $p(z_s)$, is the price such that the spot markets for both goods clear:

$$\sum_h \alpha^h \tau_{1s}^h = 0, \tag{102}$$

$$\sum_h \alpha^h \tau_{2s}^h = 0. \tag{103}$$

Hence, an attainable allocation with the spot markets is defined by adding the spot-trade constraint (101) and market-clearing constraints (102)-(103) to Definition 2.

To be more precise, an allocation is said to be *equivalent* to an attainable allocation if it is attainable and generates the same consumption allocation and market fundamental in each state s as the original attainable allocation.

Lemma 3. *For any attainable allocation* $\left(\mathbf{c}_0^h, k^h, \theta_{\ell s}^h, \tau_{\ell s}^h\right)_h$*, there exists an* ***equivalent*** *allocation* $\left(\mathbf{c}_0^h, k^h, \theta_{\ell s}'^h, \tau_{\ell s}'^h\right)_h$ *such that*

$$\tau_{\ell s}'^h = 0, \forall s, h, \ell. \tag{104}$$

Proof. Let $\left(\mathbf{c}_0^h, k^h, \theta_{\ell s}^h, \tau_{\ell s}^h\right)_h$ be an attainable allocation. We will show that we can find an equivalent allocation with no spot trade, i.e., $\tau_{\ell s}'^h = 0$. Consider the following candidate allocation (with $'$)

$$\mathbf{c}_0'^h = \mathbf{c}_0^h, \forall h, \tag{105}$$

$$\theta_{1s}'^h = \theta_{1s}^h + \tau_{1s}^h, \forall s, h, \tag{106}$$

$$\theta_{2s}'^h = \theta_{2s}^h + \tau_{2s}^h, \forall s, h. \tag{107}$$

Note that agents here acquire or issue securities on good 1 and good 2 in state s rather than waiting for trade in spot markets. The rest of the proof is similar to the proof of Lemma 2, and hence is omitted (it is available in our Working Paper version). □

Condition (104) in Lemma 3 implies that the spot markets in period 1 are redundant when all securities are allowed; that is, anything that can be done through the spot markets and one set of securities is feasible under another set of securities without spot markets. Henceforth (and previously), the ex-post spot trade transfers will be (were) set to zero, ($\tau_{\ell s}^h = 0$ as in (104)) and the spot-trade constraints (101) will be (were) neglected, unless stated otherwise.

Lemma 4. *For any attainable allocation* $\left(\mathbf{c}_0^h, k^h, \psi_{\ell s}^h, \sigma_{\ell s}^h, \nu_{\ell s}^h, \tau_{\ell s}^h\right)_h$*, there exists an* ***equivalent*** *allocation* $\left(\mathbf{c}_0^h, k^h, \psi_{\ell s}'^h, \sigma_{\ell s}'^h, \nu_{\ell s}'^h, \tau_{\ell s}'^h\right)_h$ *such that*

$$\sigma_{1s}'^h = \sigma_{2s}'^h = \nu_{1s}'^h = \nu_{2s}'^h = 0, \forall s, h. \tag{108}$$

Proof. Suppose $\left(\mathbf{c}_0^h, k^h, \psi_{\ell s}^h, \sigma_{\ell s}^h, \nu_{\ell s}^h, \tau_{\ell s}^h\right)_h$ is attainable. Consider the following alternative allocation (with $'$) $\left(\mathbf{c}_0^h, k^h, \psi_{\ell s}'^h, \sigma_{\ell s}'^h, \nu_{\ell s}'^h, \tau_{\ell s}'^h\right)_h$ such that for all h and all s

$$\sigma_{1s}'^h = \sigma_{2s}'^h = \nu_{1s}'^h = \nu_{2s}'^h = \psi_{2s}'^h = 0, \tag{109}$$

$$\psi_{1s}'^h = \left(\psi_{1s}^h + \sigma_{1s}^h + \nu_{1s}^h\right) + p(z_s)\left(\psi_{2s}^h + \sigma_{2s}^h + \nu_{2s}^h\right), \tag{110}$$

$$\tau_{1s}'^h = -p(z_s)\left(\psi_{2s}^h + \sigma_{2s}^h + \nu_{2s}^h\right) + \tau_{1s}^h, \tag{111}$$

$$\tau_{2s}'^h = \left(\psi_{2s}^h + \sigma_{2s}^h + \nu_{2s}^h\right) + \tau_{2s}^h. \tag{112}$$

Note that at the alternative allocation, agents will do in spot markets what they might have done in asset-backed security markets. In addition, with active spot markets, there is no need to trade in collateral-backed securities paying in good 2 (trade in the ones paying in numeraire good only). The rest of the proof is similar to the proof of Lemma 3, and hence is omitted. □

It is worthy of emphasis that Lemma 3 and Lemma 4 imply that the asset-backed securities that we need in this model are the ones that replicate spot markets. In other words, the asset-backed securities in this model (with tranching) are simply substitutes for spot markets. Henceforth, we let asset-backed securities play this role and shut down active trade in spot markets. The result is summarized in the following corollary.

Corollary 1. *Asset-backed securities and the spot markets are perfect substitute in this model.*

C.3 Spot Markets and Security Prices: No-Arbitrage Condition

The pyramiding mechanism puts a restriction on the prices of contracts traded within each security exchange. The ratio of the equilibrium prices of the securities in security exchange z_s in state s, $\frac{Q_{2s}}{Q_{1s}}$, must be equal to the marginal rate of substitution or the spot price in the security exchange, $p(z_s)$. Otherwise, there will be an arbitrage possibility (by keeping the collateral constraints satisfied with pyramiding). The result is summarized in the following lemma.

Lemma 5. *In a competitive equilibrium, for each s and z_s,*

$$Q_{2s} = p(z_s)Q_{1s}. \tag{113}$$

Using the no-arbitrage condition (113), the collateral constraints (95) can be rewritten as

$$Q_{2s}R_s k^h + Q_{1s}\theta_{1s}^h + Q_{2s}\theta_{2s}^h \geq 0, \forall s. \tag{114}$$

These constraints state that the value in units of good 1 at $t = 0$ of all ex ante securities held (RHS) cannot exceed the value of collateral held (LHS). These constraints are applicable when the spot markets are not available but the ex-ante asset-backed securities can be traded.

C.4 Derivation of a Competitive Equilibrium with the Externality in Environment 1

The endowment profile and the first-best allocation suggest that agent 2 would like to move resources forward from $t = 1$ to $t = 0$, and therefore will be constrained. Hence, we will assume that agents type 2 hold no collateral, i.e. $k^1 = k$ and $k^2 = 0$. We now solve for an equilibrium k. From the market clearing conditions of contracts, we can set $\theta^1_{11} = -\theta^2_{11} = \hat{\theta}$ and $\theta^1_{21} = -\theta^2_{21} = \theta$. Note that this does not mean agent 1 is demanding both securities. In addition, using the specified collateral allocation, the market fundamental in period $t = 1$ is now $z = \frac{4}{4+k}$ (the ratio of endowment of good 1 to the sum of endowment of good 2 and saving), and consequently the spot price of good 2 in period 1 is $p(z) = \left(\frac{4}{4+k}\right)^2$.

With homothetic preferences, the first-order conditions of the problem (2) for both types imply that in spot markets at date $t = 0$

$$p_0 = \left(\frac{c^1_{10}}{c^1_{20}}\right)^2 = \left(\frac{c^2_{10}}{c^2_{20}}\right)^2 = \left(\frac{4}{4-k}\right)^2. \tag{115}$$

Since agent 1's collateral constraint is not binding, the first-order conditions of her utility-maximization problem (2) with respect to θ^1_{21} and c^1_{10} lead to

$$Q_{21} = \frac{u^1_{21}}{u^1_{10}} = \left(\frac{c^1_{10}}{c^1_{21}}\right)^2, \tag{116}$$

where $u^h_{it} = \frac{\partial u^h}{\partial c_{it}}$ is the marginal utility with respect to c_{it}, and Q_{21} is the price of a security paying in good 2 in period $t = 1$. Note that we put superscript h on the utility function for clarity. Further, the first-order conditions of the consumer's problem (2) with respect to θ^1_{21} and k^1 (interior solutions) lead to

$$p_0 = Q_{21}. \tag{117}$$

Intuitively, this is the case because their payoffs are identical and both are collateralizable. Using (115) and (116), condition (117) implies that

$$\frac{c^1_{10}}{c^1_{20}} = \frac{c^1_{10}}{c^1_{21}} \Longrightarrow c^1_{20} = c^1_{21}. \tag{118}$$

That is, an unconstrained agent consumes the same amount of good 2 in both periods.

Substituting (115) and (116) into (117) gives

$$\begin{aligned}\left(\frac{4}{4-k}\right)^2 &= \left(\frac{c_{10}^1}{c_{21}^1}\right)^2;\\ \frac{4}{4-k} &= \frac{c_{10}^1}{1+k+\theta} \Longrightarrow (4-k)\,c_{10}^1 = 4+4k+4\theta, \end{aligned} \tag{119}$$

where we use $c_{21}^1 = 1+k+\theta$.

On the other hand, an agent type 2's collateral constraint is binding; with $k^2 = 0$,

$$\hat{\theta}^2 + p(z)\theta^2 = 0 \Longrightarrow -\hat{\theta} - p(z)\theta = 0 \Longrightarrow \hat{\theta} = -\left(\frac{4}{4+k}\right)^2\theta, \tag{120}$$

where the second and the last equations use $\hat{\theta}^2 = -\hat{\theta}$ and $\theta^2 = -\theta$, and $p(z) = \left(\frac{4}{4+k}\right)^2$, respectively.

The budget constraint of an agent 1 (3) can be written as

$$c_{10}^1 - 3 + p_0\left[c_{20}^1 + k - 3\right] + Q_{11}\hat{\theta} + Q_{21}\theta = 0. \tag{121}$$

A standard no-arbitrage argument (similar to the one used in Lemma 5) implies that

$$Q_{21} = p(z)Q_{11}. \tag{122}$$

It thus true from (122) that

$$Q_{11}\hat{\theta} + Q_{21}\theta = Q_{11}\hat{\theta} + Q_{11}p(z)\theta = Q_{11}\left[\hat{\theta} + p(z)\theta\right]p(z) = 0, \tag{123}$$

where the last equation follows the fact that the term in the bracket is zero, from (120). Now the LHS of the budget constraint (121) can be rewritten as

$$c_{10}^1 + p_0\left[c_{20}^1 + k - 3\right] = 3. \tag{124}$$

Using (115), we can replace c_{20}^1 by $\left(\frac{4-k}{4}\right)c_{10}^1$. Then using $p_0 = \left(\frac{4}{4-k}\right)^2$ gives

$$\begin{aligned} c_{10}^1 + \left(\frac{4}{4-k}\right)^2\left[\left(\frac{4-k}{4}\right)c_{10}^1 + k - 3\right] &= 3 \\ \Longrightarrow (4-k)\,c_{10}^1 &= \frac{3k^2 - 40k + 96}{8-k}. \end{aligned} \tag{125}$$

Substituting (119)into (125) gives

$$\frac{3k^2 - 40k + 96}{8-k} = 4 + 4\theta + 4k \Longrightarrow 4\theta + 4k = \frac{3k^2 - 36k + 64}{8-k}. \tag{126}$$

With the identical homothetic preferences, the period $t = 1$ consumption allocations must satisfy

$$z = \frac{4}{4+k} = \frac{c^1_{11}}{c^1_{21}} \Longrightarrow \frac{4}{4+k} = \frac{1+\hat{\theta}}{1+k+\theta}. \tag{127}$$

Substitute (120) into (127) gives

$$4\theta + 4k = -3k\left(\frac{4+k}{8+k}\right) + 4k. \tag{128}$$

Using (126) and (128), we have

$$\frac{3k^2 - 36k + 64}{8-k} = -3k\left(\frac{4+k}{8+k}\right) + 4k \Longrightarrow 4k^3 - 384k + 512 = 0. \tag{129}$$

There are three roots for equation (129). Using the condition that $0 \le k \le 4$, there is only one feasible solution, i.e. $k \approx 1.3595$. To sum up, the equilibrium collateral allocation is $k^1 = k = 1.3595$ and $k^2 = 0$.

C.5 Derivation of a Competitive Equilibrium with the Externality in Environment 2

First of all, the symmetry of the environment implies that the equilibrium collateral allocation is also symmetric, i.e. $k^h = k$ for all h. As a result, the price of good 2 in period $t = 0$ is given by

$$p_0 = \left(\frac{2}{2-k}\right)^2, \tag{130}$$

and the spot price of good 2 in each state s is given by

$$p_s = \left(\frac{2}{2+k}\right)^2, \forall s. \tag{131}$$

Further, the price of a (collateralized) security paying in good 2 in state s is given by

$$Q_{2s} = \max_h \left(\frac{\pi_s u^h_{2s}}{u^h_{10}}\right), \ \forall s. \tag{132}$$

The endowment structure implies that agents type 2 will have higher MRS $\frac{\pi_s u^h_{2s}}{u^h_{10}}$ in state 1, and vice versa. In addition, the structure also implies that $\theta^1_{21} = \theta^2_{22} = \theta$ and $\theta^1_{11} = \theta^2_{12} = \hat{\theta}$.

Hence, (132) can be rewritten as

$$Q_{21} = \frac{\pi_s u_{21}^2}{u_{10}^2} = \frac{1}{2}\left(\frac{2}{1+k^1+\theta_{21}^1}\right)^2 = \frac{1}{2}\left(\frac{2}{1+k+\theta}\right)^2 = \frac{1}{2}\left(\frac{2}{1+k^2+\theta_{22}^2}\right)^2 = \frac{\pi_s u_{22}^1}{u_{10}^1} = Q_{22}. \tag{133}$$

That is, the symmetry structure implies that $Q_{21} = Q_{22}$. Using the optimal conditions with respect to k^h and θ_{2s}^h, we can show that

$$p_0 = Q_{21} + Q_{22} \Longrightarrow \left(\frac{2}{2-k}\right)^2 = \left(\frac{2}{1+k+\theta}\right)^2. \tag{134}$$

Next, with the homotheticity of preferences, the ratio of consumption in each state of each agent must be equal to the market fundamental; that is,

$$\frac{1+\hat{\theta}}{1+k+\theta} = \frac{2}{2+k}. \tag{135}$$

Furthermore, the collateral constraint in state $s = 1$ of an agent type $h = 1$ is binding, i.e.

$$p_1 k - \hat{\theta} - p_1\theta = 0 \Longrightarrow \hat{\theta} = \left(\frac{2}{2+k}\right)^2 (k-\theta). \tag{136}$$

Note that the same equation can be derived from the binding collateral constraint in state $s = 2$ for an agent type $h = 2$.

We can compute a collateral equilibrium using (134), (135), and (136) to solve for $\left(k, \theta, \hat{\theta}\right)$. We can rewrite (134) as

$$2 - k = 1 + k + \theta \Longrightarrow \theta = 1 - 2k. \tag{137}$$

In addition, Substituting (136) into (135) gives

$$1 + \left(\frac{2}{2+k}\right)^2 (k-\theta) = \left(\frac{2}{2+k}\right)(1+k+\theta). \tag{138}$$

Then, substituting (137) into (138) will give

$$\begin{aligned} 1 + \left(\frac{2}{2+k}\right)^2 (k-1+2k) &= \left(\frac{2}{2+k}\right)(1+k+1-2k) \\ &\Longrightarrow 3k^2 + 16k - 8 = 0. \end{aligned} \tag{139}$$

The unique feasible (positive) solution to the above quadratic equation is $k \approx 0.4603$.

C.6 Derivation of a Competitive Equilibrium with the Externality in Environment 3

We restrict our attention to a symmetric allocation of each type. Using Lemma , we assume that all constrained agents hold no collateral, i.e., $k^h = 0$ for $h = 2, 3$. Let $k^1 = k$.

First, the first-order conditions of the consumer's problem (2) result in

$$\frac{c_{10}^1}{c_{20}^1} = \frac{c_{10}^2}{c_{20}^2} = \frac{c_{10}^3}{c_{20}^3} = \frac{12.5}{12.5 - k}. \tag{140}$$

From the endowment profile, it is clear that an agent 1 will not be constrained. The first-order conditions of the consumer's problem (2) with respect to θ_{21}^1 and c_{10}^1 lead to

$$\frac{u_{21}^1}{u_{10}^1} = Q_{21}. \tag{141}$$

Further, the first-order conditions of the consumer's problem (2) with respect to θ_{21}^1 and k^1 (interior solutions) lead to

$$p_0 = Q_{21}. \tag{142}$$

Combining (141), (142) and the utility function (26), gives

$$p_0 = \left(\frac{12.5}{12.5 - k}\right)^2 \quad = \quad Q_{21} = \frac{u_{21}^1}{u_{10}^1} = \left(\frac{c_{10}^1}{c_{21}^1}\right)^2. \tag{143}$$

This implies that

$$\frac{12.5}{12.5 - k} \quad = \quad \frac{c_{10}^1}{c_{21}^1} = \frac{c_{10}^1}{0.5 + k + \theta_{21}^1} \Longrightarrow (12.5 - k)\, c_{10}^1 = 12.5\left(0.5 + k + \theta_{21}^1\right), \tag{144}$$

where we use $c_{21}^1 = 0.5 + k + \theta_{21}^1$.

In addition, the market fundamental in period $t = 1$ is $z = \frac{12.5}{12.5+k}$, and consequently the spot price of good 2 in period $t = 1$ is $\left(\frac{12.5}{12.5+k}\right)^2$. The bindingness of the collateral constraints of agent 2 and agent 3, combining with the market-clearing conditions of securities, imply that

$$\theta_{11}^1 = -\left(\frac{12.5}{12.5 + k}\right)^2 \theta_{21}^1. \tag{145}$$

A standard no-arbitrage argument (similar to the one used in Lemma 5) implies that

$$Q_{21} = p(z) Q_{11}, \tag{146}$$

which can be used to show that

$$Q_{11}\theta_{11}^1 + Q_{21}\theta_{21}^1 = Q_{11}\theta_{11}^1 + Q_{11}p(z)\theta_{21}^1 = Q_{11}\left[\theta_{11}^1 + p(z)\theta_{21}^1\right]p(z) = 0, \tag{147}$$

where the last equation follows the bindingness of the collateral constraints of agent 2 and agent 3, combining with the market-clearing conditions of securities. The budget constraint of an agent 1 (3) can be written as

$$c_{10}^1 - e_{10}^1 + p_0\left[c_{20}^1 + k - e_{20}^1\right] = 0. \tag{148}$$

Substituting (140) and (143) into (148), we have

$$(12.5 - k)\, c_{10}^1 = \frac{12.5^2\left(e_{20}^1 - k\right) + e_{10}^1\left(12.5 - k\right)^2}{25 - k}. \tag{149}$$

Substituting (144) into (149), we have

$$12.5\left(0.5 + k + \theta_{21}^1\right) = \frac{12.5^2\left(e_{20}^1 - k\right) + e_{10}^1\left(12.5 - k\right)^2}{25 - k}. \tag{150}$$

With the identical homothetic preferences, the period $t = 1$ consumption allocations must satisfy

$$z = \frac{12.5}{12.5 + k} = \frac{c_{11}^1}{c_{21}^1} \Longrightarrow \frac{12.5}{12.5 + k} = \frac{0.5 + \theta_{11}^1}{0.5 + k + \theta_{21}^1}, \tag{151}$$

where the equality follows (145). This can be rewritten as

$$12.5\left(0.5 + k + \theta_{21}^1\right) = (12.5 + k)\left(0.5 - \left(\frac{12.5}{12.5 + k}\right)^2 \theta_{21}^1\right). \tag{152}$$

Solving (150) and (152) for k and θ_{21}^1, with $e_{10}^1 = 4.2631$ and $e_{20}^1 = 11.5$, gives one feasible solution ($0 \le k \le 12.5$) $k = 7.2836, \theta_{21}^1 = -4.2849$. To sum up, the competitive collateral equilibrium allocation is $k^1 = k = 7.2836$, and $k^2 = k^3 = 0$.

C.7 Source of Inefficiency in the Incomplete Markets Example

Proposition 3. *The competitive equilibrium with exogenous security markets is (constrained) efficient if and only if the equilibrium allocation* $\left(\mathbf{c}^h, \theta^h, \tau^h, \mathbf{y}^h\right)$ *is first-best optimal or the spot price is independent of security positions, i.e.,* $\frac{\partial p_s}{\partial \theta_j^h} = 0$ *for every state* s*, every security* j *and every agent of type* h*.*

Proof. We begin the proof by deriving the necessary and sufficient conditions for the first-best optimality. The social planner's problem for the first-best optimality is as follows:

Program 3.

$$\max_{\left(\theta_{i0}^h,\theta_{is}^h\right)_{i,s,h}} u^1\left(e_{10}^1+\theta_{10}^1, e_{20}^1+\theta_{20}^1\right)+\beta\sum_s \pi_s u^1\left(e_{1s}^1+\theta_{1s}^1, e_{2s}^1+\theta_{2s}^1\right) \tag{153}$$

subject to the participation constraints and the resource constraints, respectively,

$$\begin{aligned} u^h\left(e_{10}^h+\theta_{10}^h, e_{20}^h+\theta_{20}^h\right)+\beta\sum_s \pi_s u^h\left(e_{1s}^h+\theta_{1s}^h, e_{2s}^h+\theta_{2s}^h\right) &\geq \overline{\mathscr{U}}^h, \text{for } h=2,...,H, \\ \sum_h \alpha^h\theta_{is}^h &= 0, \text{for } i=1,2; s=0,1,...,S \end{aligned}$$

Lemma 6. *The necessary and sufficient conditions for the first-best optimality are as follows:*

$$\frac{\gamma_u^h u_{i0}^h}{\alpha^h} = \frac{\gamma_u^{\tilde{h}} u_{i0}^{\tilde{h}}}{\alpha^{\tilde{h}}}, \forall h, \tilde{h}=1,...,H; i=1,2 \tag{154}$$

$$\frac{\gamma_u^h \beta\pi_s u_{is}^h}{\alpha^h} = \frac{\gamma_u^{\tilde{h}} \beta\pi_s u_{is}^{\tilde{h}}}{\alpha^{\tilde{h}}}, \forall h, \tilde{h}=1,...,H; i=1,2, s=1,...,S, \tag{155}$$

where γ_u^h is the Lagrange multipliers for the participation constraints for h (normalize by setting $\gamma_u^1=1$) and $u_{is}^h=\frac{\partial u^h}{\partial c_{is}^h}$ is the marginal utility of an agent of type h with respect to c_{is}.

We now consider the following social planner's problem for the economy with exogenous security markets.

Program 4.

$$\max_{\left(\theta_{10}^h,\theta_{20}^h,\theta_j^h,\tau_{1s}^h,\tau_{2s}^h\right)_h} u^1\left(e_{10}^1+\theta_{10}^1, e_{20}^1+\theta_{20}^1\right)+\beta\sum_s \pi_s u^1\left(e_{1s}^1+\theta_s^1+\tau_{1s}^1, e_{2s}^1+\tau_{2s}^1\right) \tag{156}$$

subject to the participation constraints, the resource constraints, and the obstacle-to-trade constraints, respectively,

$$u^h\left(e_{10}^h+\theta_{10}^h, e_{20}^h+\theta_{20}^h\right)+\beta\sum_s \pi_s u^h\left(e_{1s}^h+\sum_j D_{js}\theta_j^h+\tau_{1s}^h, e_{2s}^h+\tau_{2s}^h\right) \geq \overline{\mathscr{U}}^h, \forall h, \tag{157}$$

$$\sum_h \alpha^h\theta_{i0}^h = 0, \forall i, \tag{158}$$

$$\sum_h \alpha^h\theta_j^h = 0, \forall j, \tag{159}$$

$$\sum_h \alpha^h\tau_{1s}^h = 0, \forall s, \tag{160}$$

$$\tau_{1s}^h+\tilde{p}_s\left(\theta_s^1,...,\theta_s^H\right)\tau_{2s}^h = 0, \forall s, h. \tag{161}$$

Note that the resource (market-clearing) constraints for τ^h_{2s} are omitted due to Walras law. A solution to this social planner's problem is called a constrained optimal allocation.

The first order conditions for $\theta^h_{10}, \theta^h_{20}, \tau^h_{1s}, \tau^h_{2s}, \theta^h_j$ are as follows:

$$\gamma^h_u \beta \pi_s u^h_{10} + \alpha^h \mu^\theta_{10} = 0, \tag{162}$$

$$\gamma^h_u \beta \pi_s u^h_{20} + \alpha^h \mu^\theta_{20} = 0, \tag{163}$$

$$\gamma^h_u \beta \pi_s u^h_{1s} + \alpha^h \mu^\tau_{1s} + \gamma^h_s = 0, \forall s = 1, ..., S, \tag{164}$$

$$\gamma^h_u \beta \pi_s u^h_{2s} + \tilde{p}_s \gamma^h_s = 0, \forall s = 1, ..., S, \tag{165}$$

$$\gamma^h_u \beta \sum_s \pi_s u^h_{1s} D_{js} + \alpha^h \mu^\theta_j + \sum_s \frac{\partial \tilde{p}_s}{\partial \theta^h_j} \sum_{\tilde{h}} \gamma^{\tilde{h}}_s \tau^{\tilde{h}}_{2s} = 0, \forall j = 1, ..., J, \tag{166}$$

where $\gamma^h_s, \gamma^h_s, \mu^\tau_s, \mu^\theta_j$ are the Lagrange multipliers for the obstacle to trade or spot-market constraints in state s, for the participation constraints for h (normalize by setting $\gamma^1_u = 1$), for the resource constraints for τ^h_{1s}, and for the resource constraints for θ^h_j. Note that $u^h_{is} = \frac{\partial u^h}{\partial c^h_{is}}$.

The proof is divided into two parts as follows:

(i) ($\Leftarrow$) We now show that an allocation that satisfies the necessary and sufficient conditions for the first-best optimality (154)-(155) must satisfies the first order conditions (162)-(166). It is not difficult to see that this will be the case if the externality term, the last term of (166), is vanished, i.e.,

$$\sum_s \frac{\partial \tilde{p}_s}{\partial \theta^h_j} \sum_{\tilde{h}} \gamma^{\tilde{h}}_s \tau^{\tilde{h}}_{2s} = 0 \tag{167}$$

It is obvious that if the spot price is independent of security positions, i.e., $\frac{\partial \tilde{p}_s}{\partial \theta^h_j} = 0$ for every state s, every security j and every agent of type h, then condition (167) holds.

We now need to show that if the constrained optimal allocation is first-best optimal, then the no-externality condition (167) must hold. Since the allocation is first- best optimal, it must satisfies conditions (154) and (155), which imply that $\left(\frac{\gamma^{\tilde{h}}_s}{\alpha^{\tilde{h}}}\right)$ must be constant across agents, i..e, for each s

$$\frac{\gamma^h_s}{\alpha^h} = \frac{\gamma^{\tilde{h}}_s}{\alpha^{\tilde{h}}} = \Gamma_s, \forall h, \tilde{h}. \tag{168}$$

Using these conditions, we can then show that

$$\sum_s \frac{\partial \tilde{p}_s}{\partial \theta_j^h} \sum_{\tilde{h}} \gamma_s^{\tilde{h}} \tau_{2s}^{\tilde{h}} = \sum_s \frac{\partial \tilde{p}_s}{\partial \theta_j^h} \sum_{\tilde{h}} \left(\frac{\gamma_s^{\tilde{h}}}{\alpha^{\tilde{h}}} \right) \alpha^{\tilde{h}} \tau_{2s}^{\tilde{h}} \tag{169}$$

$$= \sum_s \frac{\partial \tilde{p}_s}{\partial \theta_j^h} \sum_{\tilde{h}} \mathit{\Gamma}_s \alpha^{\tilde{h}} \tau_{2s}^{\tilde{h}} = \sum_s \frac{\partial \tilde{p}_s}{\partial \theta_j^h} \mathit{\Gamma}_s \sum_{\tilde{h}} \alpha^{\tilde{h}} \tau_{2s}^{\tilde{h}} = 0, \tag{170}$$

where the last equation results from the resource constraints for τ_{2s}^h. This proves that the no-externality condition (167) holds. To sum up, we prove that there is no externality *if* the constrained optimal allocation is first-best optimal or the spot price is independent of security positions, i.e., $\frac{\partial \tilde{p}_s}{\partial \theta_j^h} = 0$ for every state s, every security j and every agent of type h.

(ii) ($\Rightarrow$) Unfortunately, we cannot generally prove the reversed statement but, as shown in Geanakoplos and Polemarchakis (1986), it is true generically (it is true except for some unlikely cases). The key idea is that the indirect price effects could be canceling each other out only if the equilibrium allocation is first-best optimal in most cases. But this does not happen generally.

□