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Generalized Welfare Gains from Trade Formulas

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Generalized Welfare Gains From Trade Formulas*

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Abstract

This paper develops a novel approach to derive welfare gains from trade formulas applicable to a wide range of trade models. First, I derive a general formula for classical trade models based on the formula established by Arkolakis et al. (2012), henceforth referred to as "ACR." This new formula incorporates a nuanced consideration of the production share and elasticity of substitution between outputs, offering a refined formula for assessing the welfare gains associated with movements along the production possibility frontier. Second, I use this new approach to generalize the ACR formula. This new derivation broadens the applicability of the ACR formula, extending its relevance to a broader class of trade models than previously considered.

Keywords: trade theory, gains from trade, classical trade models, ACR formula
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1 Introduction

In a notable contribution to the field of international economics, Arkolakis, Costinot, and Rodríguez-Clare (2012)—hereafter referred to as ACR—show that in the class of trade models that satisfy three macro-level restrictions: (i) trades are balanced, (ii) aggregate profits are a fixed proportion of total income, and (iii) the demand system follows constant-elasticity-of-substitution (CES) preferences, aggregate welfare gains from trade can be captured by two pivotal statistics: (i) the expenditure share of domestic goods, denoted by λ, and (ii) the trade elasticity with respect to variable trade costs, denoted by ε. ACR elegantly encapsulate the idea that the change in welfare associated with a change in trade costs can be measured by

$$d\ln W = \frac{d\ln\lambda}{\varepsilon}.$$

Following ACR's groundbreaking work, numerous studies have emerged, broadening the original framework's reach and depth. These extensions explore diverse model features, ranging from heterogeneous firm models (Melitz and Redding 2015) to those with variable markups (Arkolakis et al., 2019). This paper revisits and extends the welfare gain formula of Arkolakis et al. (2012) in two fundamental ways. First, it derives a welfare gain from trade formula for classical trade models. Second, it examines the possibility of deriving a welfare gain formula without relying strictly on the three macro-level restrictions identified by ACR.

First, the paper introduces a general welfare gain formula to classical trade models, particularly general-equilibrium models with a production possibility frontier and indifference curves. I propose a general welfare formula that captures gains from trade via adjustments in the production and consumption points. Similarly to the ACR formula, the change in production along the surface of the production possibility frontier can be measured by two pivotal statistics: the production share and the elasticity of output substitution. To be precise, under certain conditions, the general welfare gain formula for classical trade models is

$$d\ln W = \frac{d\ln\lambda_1}{1-\overline{\sigma}} - \frac{d\ln s_1}{\overline{\eta}+1},$$

where λ_1 and s_1 are the expenditure share and production share of the same good, and $\overline{\sigma}$ and $\overline{\eta}$ represent the elasticities of substitution in consumption and production, respectively.

Second, this study revisits the standard Armington model and general preferences. I use Roy's identity and the concept of elasticity of substitution to develop a generalized ACR formula. This approach shows that welfare change due to a price change can be approximated by the corresponding expenditure share without assuming the CES preferences. This paper then shows that the second key equation in ACR's methodology is a direct result of the definition of the elasticity of substitution. By relaxing the third restriction in ACR, the elasticity of substitution may vary with quantities under non-CES preferences. Combining these two new equations yields a generalized formula for welfare gains from trade.

The novel general formula for welfare gains from trade incorporates terms for the relationship between welfare and income, acknowledges varying elasticities of substitution across goods, and accounts for potential changes in total income. Under certain less restrictive assumptions than ACR's three macro-level restrictions, this formula could be simplified to the well-known ACR formula. Therefore, this new derivation suggests that the ACR formula applies to a broader class of trade models than previously recognized.

Furthermore, the paper discusses how to adjust the welfare gain formula for scenarios involving trade imbalances and aggregate profits. This adjustment is straightforward, requiring only data on changes in trade imbalances and aggregate profits, and their proportions in total income. I illustrate how the new derivation can be easily applied to the Melitz model and other trade models. The main idea lies in leveraging observable data (e.g., expenditure shares and elasticities) to deduce the changes in prices and income that eventually influence welfare gains.

In sum, this paper not only revisits classical trade models and extends a welfare gain from trade formula but also offers a more comprehensive understanding of the ACR formula.

The remainder of the paper is structured as follows. Section 2 describes a classical trade model and presents a derivation for a general formula. Section 3 reviews the basic model in Arkolakis et al. (2012) and their proof. Section 4 introduces a new derivation and applications. Section 5 concludes.

2 A General Formula for Classical Trade Models

This section revisits a classical trade model and presents a derivation for a formula for welfare gains from trade.[1]

2.1 Model Setup

This section explores a classical international trade model, where a country, hereafter "Home," transitions from autarky to engaging in the world market.

Consider Home's economy, encompassing n goods with the $n \times 1$ price vector $\boldsymbol{p} \equiv \{p_i\}$, and a single production factor called labor.[2] The output vector $\boldsymbol{y} = \{y_i\}$ denotes Home's production, where y_i corresponds to the output of good i. The production possibility frontier is depicted by $F(\boldsymbol{y}; L) = 0$, adhering to the conventional economic assumptions of twice differentiability, diminishing marginal returns, and constant returns to scale.

Home's income, $R(\boldsymbol{p}) = \boldsymbol{p} \cdot \boldsymbol{y}(\boldsymbol{p})$, is a function of the production output under the prevailing price vector $\boldsymbol{p}$. The representative agent selects a consumption vector $\boldsymbol{x} = \{x_i\}$, where x_i is the consumption of good i, to maximize the utility function $U(\boldsymbol{x})$, constrained by the price vector $\boldsymbol{p}$ and income $R(\boldsymbol{p})$. The optimal consumption is thus $\boldsymbol{x}(\boldsymbol{p}, R(\boldsymbol{p}))$. Home's welfare is calculated by

[1] The derivation is based on Dixit and Norman (1980).

[2] The model can have more than one input, but a restriction on the production function is required to establish existence and uniqueness and to avoid reversal in production.

the indirect utility function $V(\boldsymbol{p}, R(\boldsymbol{p})) \equiv U(\boldsymbol{x}(\boldsymbol{p}, R(\boldsymbol{p})))$, which is a function of price vector $\boldsymbol{p}$ and income $R(\boldsymbol{p})$.

In autarky, the market clearing conditions determine the autarky price vector, denoted by $\boldsymbol{p}^A$, such that that $\boldsymbol{p}^A \cdot \left(\boldsymbol{x}\left(\boldsymbol{p}^A\right) - \boldsymbol{y}\left(\boldsymbol{p}^A\right)\right) = 0$. When Home is open to trade with the world, it is facing a new world price vector $\boldsymbol{p}$. It is assumed that the prices are strictly positive and an equilibrium always exists and is unique.

I define $s_i = p_i y_i / \left(\sum_{j=1}^{n} p_j y_j\right)$ as the share of Home's total income that comes from the production of good i, and $\lambda_i = p_i x_i / \left(\sum_{j=1}^{n} p_j x_j\right)$ as the expenditure share of Home's consumption of good i.

Let $\eta_{ij}(\boldsymbol{y})$ be the elasticity of output substitution between goods 1 and i from a change in the relative price p_j/p_1 evaluated at the output vector $\boldsymbol{y}$ as

$$\eta_{ij}(\boldsymbol{y}) = \frac{d\ln(y_i/y_1)}{d\ln\left(p_j/p_1\right)}. \tag{1}$$

This elasticity measures the percentage changes in production as the price plane rotates along the surface of the production possibility frontier (PPF). The elasticity may vary along the PPF.

Similarly, the elasticity of substitution between goods 1 and i from a change in the relative price p_j/p_1 evaluated at the consumption vector $\boldsymbol{x}$ is denoted by

$$\sigma_{ij}(\boldsymbol{x}) = -\frac{d\ln(x_i/x_1)}{d\ln\left(p_j/p_1\right)}. \tag{2}$$

The elasticity of substitution is allowed to vary and depend on the consumption. Without loss of generality, good 1 is treated as a numeraire, and its price is normalized to one.

I define three $(n-1) \times 1$ vectors: $\boldsymbol{DS}$, $\boldsymbol{DL}$, and $\boldsymbol{DP}$. $\boldsymbol{DS}$ captures relative changes in production shares, where its jth element is $d\ln\left(s_{j+1}/s_1\right)$. $\boldsymbol{DL}$ measures relative changes in consumption shares, where its jth element is $d\ln\left(\lambda_{j+1}/\lambda_1\right)$. $\boldsymbol{DP}$ presents price change and its jth element is $d\ln\left(p_{j+1}/p_1\right)$.

These three vectors are interconnected through linear relationships, defined by matrices $\boldsymbol{H}$ and $\boldsymbol{\Lambda}$:

$$\boldsymbol{DS} = \boldsymbol{H} \cdot \boldsymbol{DP} \tag{3}$$

$$\boldsymbol{DL} = \boldsymbol{\Lambda} \cdot \boldsymbol{DP}, \tag{4}$$

where $\boldsymbol{H}$ is a square matrix with dimensions $(n-1) \times (n-1)$ and its ijth element represents $d\ln(s_{i+1}/s_1)/d\ln\left(p_{j+1}/p_1\right)$, and $\boldsymbol{\Lambda}$ is a square matrix with dimensions $(n-1) \times (n-1)$ and its ijth element represents $d\ln\left(\lambda_{j+1}/\lambda_1\right)/d\ln\left(p_{j+1}/p_1\right)$.

Finally, I define $\boldsymbol{s} = \{s_{i+1}\}$ as the $(n-1) \times 1$ vector of production shares and $\boldsymbol{\lambda} = \{\lambda_{i+1}\}$ as the $(n-1) \times 1$ vector of the expenditure shares.

2.2 Proof

When Home experiences a change in price p_j, the associated welfare change is equal to

$$\frac{\partial V\left(\boldsymbol{p}, R\left(\boldsymbol{p}\right)\right)}{\partial p_j} = \left(-x_j + y_j\right)\frac{\partial V}{\partial R},$$

where $\partial V/\partial R$ is the marginal utility of additional income.

This equation states that the welfare gain from price change dp_j depends on $\left(-x_j + y_j\right)dp_j$, where the term $\left(-x_j + y_j\right)$ is the net export of good j. Therefore, Home experiences welfare gains if (i) Home is an exporter of good j and price p_j increases, or (ii) Home is an importer of good j and price p_j decreases. In contrast, Home receives welfare losses if (i) Home is an exporter of good j and price p_j decreases, or (ii) Home is an importer of good j and price p_j increases. The size of the welfare change in terms of money depends on the magnitudes of the net export/import and the price change. This welfare change is converted from monetary value to utility units using a money metric denoted by $\partial V/\partial R$.

From the indirect utility function, welfare gains that arise from a change in world prices can be calculated by

$$dV = \sum_{j=1}^{n}\frac{\partial V}{\partial p_j}dp_j = \left(\frac{\partial V}{\partial R}\right)\sum_{j=1}^{n}\left(-x_j + y_j\right)dp_j. \tag{5}$$

From the definitions of expenditure and production shares, equation (5) can be simplified to

$$d\text{ln}V = \left(\frac{R}{V}\frac{\partial V}{\partial R}\right)\left[-\left(\sum_{j=1}^{n}\lambda_j d\text{ln}p_j\right) + \left(\sum_{j=1}^{n}s_j d\text{ln}p_j\right)\right]. \tag{6}$$

From equations (3) and (4), we can use production shares and expenditure shares to infer the price changes as

$$\boldsymbol{DP} = \boldsymbol{H}^{-1}\cdot\boldsymbol{DS}$$
$$\boldsymbol{DP} = \boldsymbol{\Lambda}^{-1}\cdot\boldsymbol{DL}.$$

By choice of numeraire, we have that $d\text{ln}\left(p_1\right) = 0$, and equation (6) can be simplified to the formula in Proposition 1.

Proposition 1. *Welfare gains from trade can be measured by*

$$dlnV = \left(\frac{R}{V}\frac{\partial V}{\partial R}\right)\left[-\left(\boldsymbol{\lambda}\cdot\boldsymbol{\Lambda}^{-1}\cdot\boldsymbol{DL}\right) + \boldsymbol{s}\cdot\boldsymbol{H}^{-1}\cdot\boldsymbol{DS}\right]. \tag{7}$$

This proposition suggests that the welfare change from price changes can be inferred through the elasticities of substitution and the respective changes in production and consumption. Essentially, each variation in price is linked to and can be traced through these changes in expenditure shares alongside the elasticity of substitution. The aggregate effect on welfare is essentially a weighted average of all these individual price changes. However, these calculations are made in nominal terms. Therefore, we need to apply a money metric to convert the aggregate effect to a unit of utility.

The formula includes a novel component representing the change in production, a feature not present in ACR. This distinction arises because, in the new-trade model, total income remains fixed due to the constant supply of labor and a normalized wage.

There are two observations. First, there is the money metric term $\left(\frac{R}{V}\frac{\partial V}{\partial R}\right)$ that measures the value of changes in income in terms of welfare. Under the assumption of homothetic preferences, where welfare grows linearly in income, the money metric term is equal to one. Second, in the absence of a specific assumption about the elasticities of substitution, aggregating all terms in the summation is challenging.

In the context of constant elasticity of substitution (CES) preferences and CES production functions, the inverse matrices $\boldsymbol{H}^{-1}$ and $\boldsymbol{\Lambda}^{-1}$ can be simplified to

$$\boldsymbol{H}^{-1} = \frac{1}{1-\overline{\sigma}} I_{(n-1)}$$
$$\boldsymbol{\Lambda}^{-1} = \frac{1}{\overline{\eta}+1} I_{(n-1)},$$

where $\overline{\sigma}$ is the elasticity of substitution, $\overline{\eta}$ is the elasticity of output substitution, and $I_{(n-1)}$ denotes the $(n-1) \times (n-1)$ identity matrix.

Proposition 2 lays out the specific conditions under which the welfare formula in Proposition 1 can be simplified.

Proposition 2. *Under constant elasticity of substitution preferences and constant elasticity of substitution production technology, the welfare gains from trade can be measured by*

$$dlnV = \frac{dln\lambda_1}{1-\overline{\sigma}} - \frac{dlns_1}{\overline{\eta}+1}, \tag{8}$$

where $\overline{\sigma}$ is the elasticity of substitution and $\overline{\eta}$ is the elasticity of output substitution.

This formula is comparable to the main result in ACR. This new general formula captures the changes in consumption and production points using the expenditure share and the production share of the numeraire good. The first term is related to the adjustment of consumption toward cheaper imported goods, while the second term is related to the reallocation of inputs toward the production of goods in which Home has a comparative advantage.

2.3 Example

This subsection provides an example of the formula in Proposition 2.

Let us examine a classic trade model encompassing two goods. The production functions are

$y_i = \sqrt{l_i}$ with the input constraint $l_1 + l_2 = \bar{l}$. The utility function is $U = \left(x_1^{\frac{\sigma-1}{\sigma}} + x_2^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$, where $\sigma > 1$ represents the elasticity of substitution between goods. Under prevailing prices p_1 and p_2, the supply functions and total income are

$$y_i = p_i \sqrt{\frac{\bar{l}}{p_1^2 + p_2^2}},$$
$$R = p_1 y_1 + p_2 y_2 = \sqrt{\left(p_1^2 + p_2^2\right) \bar{l}}.$$

In this case, the elasticity of substitution of output between goods 1 and i is denoted as

$$\eta_i(\boldsymbol{y}) = \frac{d\ln(y_i/y_1)}{d\ln(p_i/p_1)} = 1.$$

The demand functions are

$$x_i = \frac{p^{-\sigma}}{p_1^{1-\sigma} + p_2^{1-\sigma}} R.$$

Home's welfare is

$$V = \frac{\sqrt{\left(p_1^2 + p_2^2\right) \bar{l}}}{\left(p_1^{1-\sigma} + p_2^{1-\sigma}\right)^{\frac{1}{1-\sigma}}}.$$

Suppose that good 1 is treated as numeraire. The price p_1 is normalized to one, and let $p = p_2/p_1$ be the relative price. Home's welfare is simplified to

$$V = \frac{\sqrt{(1 + p^2)\bar{l}}}{(1 + p^{1-\sigma})^{\frac{1}{1-\sigma}}}.$$

Home's welfare change is

$$d\ln V = \left(-\frac{p^{1-\sigma}}{1 + p^{1-\sigma}} + \frac{p^2}{1 + p^2} \right) d\ln p.$$

Furthermore, the consumption and production shares are represented as

$$\lambda_1 = \frac{1}{1 + p^{1-\sigma}},$$
$$s_1 = \frac{1}{1 + p^2}.$$

The changes in these shares are calculated as

$$d\ln\lambda_1 = -\frac{(1-\sigma)\,p^{1-\sigma}}{1+p^{1-\sigma}}d\ln p,$$
$$d\ln s_1 = -\frac{2p^2}{1+p^2}d\ln p.$$

Therefore, it can be demonstrated that the welfare change in Home's economy, $d\ln V$, aligns with the formula

$$d\ln V = \frac{d\ln\lambda_1}{1-\overline{\sigma}} - \frac{d\ln s_1}{\overline{\eta}+1}.$$

3 A Review of ACR

This section briefly describes the basic model in Arkolakis et al. (2012) to highlight the assumptions in their proof.

3.1 Model Setup

Arkolakis et al. (2012) start with the Armington model of trade. There are n countries. Each country produces one differentiated good, using labor as the only input. The supply of labor is fixed at L_i. The preferences are

$$U_j = \left[\sum_{i=1}^{n} q_{ij}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{9}$$

where q_{ij} is the amount of country i's good consumed by country j, and $\sigma > 1$ is the (constant) elasticity of substitution. The price index is given by

$$P_j = \left[\sum_{i=1}^{n} \left(w_i\tau_{ij}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}}, \tag{10}$$

where w_i is the wage in country i and $\tau_{ij} > 1$ is the iceberg trade costs associated with shipping goods from country i to country j. The value of trade flows from country i to country j is

$$X_{ij} = \left(\frac{w_i\tau_{ij}}{P_j}\right)^{1-\sigma} Y_j, \tag{11}$$

where Y_j is the total income of country j. Let $\lambda_{ij} = X_{ij}/Y_j$ denote country j's expenditure share on the goods from country i. We have the identity $\sum_{i=1}^{n}\lambda_{ij} = 1$, which leads to $\sum_{i=1}^{n}\lambda_{ij}d\ln\lambda_{ij} = 0$. The budget constraint is $Y_j = w_jL_j$.

3.2 The Proof in ACR

I assume that Home's wage is a numeraire. The budget constraint implies that $d\ln Y_j = d\ln w_j + d\ln L_j = 0$ due to the choice of numeraire and the fixed supply of labor.

Equations (10) and (11) imply that

$$d\ln W_j = -\sum_{i=1}^{n} \lambda_{ij} \left(d\ln w_i + d\ln \tau_{ij} \right). \tag{12}$$

Equation (11) implies that

$$d\ln \lambda_{ij} - d\ln \lambda_{jj} = (1-\sigma) \left(d\ln w_i + d\ln \tau_{ij} \right). \tag{13}$$

Equations (12) and (13) conclude that

$$d\ln W_j = -\sum_{i=1}^{n} \lambda_{ij} \frac{\left(d\ln \lambda_{ij} - d\ln \lambda_{jj} \right)}{(1-\sigma)} = \frac{d\ln \lambda_{jj}}{(1-\sigma)}. \tag{14}$$

Arkolakis et al. (2012), show that the same welfare gain formula works in trade models that satisfy three macro-level restrictions: (A1) trade in goods is balanced; (A2) aggregate profits are a constant share of GDP; and (A3) the import demand system is CES.

4 A Generalized Formula

This section describes the parts of the model that are more general than the basic model in ACR.

4.1 Setup

The analysis starts with the Armington model of international trade. There are n countries, with each country specializing in the production of one unique, differentiated good. Labor is the only input used for production in these countries, and the supply of labor in each country is fixed, denoted as L_i. The budget constraint is

$$Y_j = w_j L_j + T_j, \tag{15}$$

where $w_j L_j$ is the labor income from wage w_j and T_j represents the (exogenous) net transfer to country j with the condition $\sum_{j=1}^{n} T_j = 0$. This net transfer can be interpreted as any factor that allows country j to consume more than what it earns through its labor income.

The preferences of country j are

$$U_j = U\left(q_{1j}, ..., q_{nj} \right), \tag{16}$$

where q_{ij} is country j's consumption of goods from country i.

The associated indirect utility function is

$$V_j = V\left(p_{1j}, ..., p_{nj}, Y_j\right), \tag{17}$$

where $p_{ij} = w_i \tau_{ij}$ is the price of country i's good sold in country j. Let $\lambda_{ij} = p_{ij} q_{ij} / Y_j$ be country j's expenditure share on the goods from country i.

Let σ_{ik} be the elasticity of substitution between goods from countries i and j from a change in the relative price price p_{kj}/p_{jj}, defined as

$$\sigma_{ik}\left(\boldsymbol{q}_j\right) = -\frac{d\ln\left(q_{ij}/q_{jj}\right)}{d\ln\left(p_{kj}/p_{jj}\right)}. \tag{18}$$

The elasticity of substitution is a function of the consumption vector $\boldsymbol{q}_j$, because it may vary based on consumption bundles.

I define three $(n-1) \times 1$ vectors: $\boldsymbol{DL}$, $\boldsymbol{DP}$, and $\boldsymbol{\lambda}$. The vector $\boldsymbol{DL}$ includes $(n-1)$ elements of $d\ln\left(\lambda_{ij}/\lambda_{jj}\right)$, where $i = 1, \ldots, n$ and $i \neq j$. The vector $\boldsymbol{DP}$ includes $(n-1)$ elements of $d\ln\left(p_{ij}/p_{jj}\right)$, where $i = 1, \ldots, n$ and $i \neq j$. Finally, The vector $\boldsymbol{\lambda}$ includes $(n-1)$ elements of λ_{ij}, where $i = 1, \ldots, n$ and $i \neq j$.

The linear relationship between $\boldsymbol{DL}$ and $\boldsymbol{DP}$ is defined by matrix $\boldsymbol{\Lambda}$:

$$\boldsymbol{DL} = \boldsymbol{\Lambda} \cdot \boldsymbol{DP}, \tag{19}$$

where $\boldsymbol{\Lambda}$ is a square matrix with dimensions $(n-1) \times (n-1)$.

The main differences from the basic model in ACR are three features. First, the preferences are not assumed to be CES. Second, the elasticity of substitution is allowed to vary based on consumption bundles. Third, total income can be different from labor income.

4.2 A New Derivation

I will derive equations that are comparable to equations (12) and (13).

Assuming that Home's wage is numeraire. From $p_{jj} = w_j$, we have $d\ln p_{jj} = d\ln w_j = 0$. Because of the labor supply is fixed and the wage w_j is normalized to one due to the choice of numeraire, the change in labor income is zero, i.e., $d\ln w_j + d\ln L_j = 0$, and the budget constraint in equation (15) implies that

$$d\ln Y_j = \left(1 - \gamma_j\right)\left(d\ln w_j + d\ln L_j\right) + \gamma_j d\ln T_j = \gamma_j d\ln T_j, \tag{20}$$

where $\gamma_j = T_j / Y_j$ is the share of country j's net transfer in its total income.

From the indirect utility function in equation (17), welfare gains can be calculated by

$$dV_j = \sum_{i=1}^{n} \frac{\partial V_j}{\partial p_{ij}} dp_{ij} + \frac{\partial V_j}{\partial Y_j} dY_j. \tag{21}$$

Using $\frac{\partial V_j}{\partial p_{ij}} = -q_i \frac{\partial V_j}{\partial Y_j}$ from Roy's identity and $d\ln Y_j = \gamma_j d\ln T_j$ in equation (20), the welfare gains can be simplified to

$$dV_j = \frac{\partial V_j}{\partial Y_j} \sum_{i=1}^{n} (-q_i)\, dp_{ij} + \frac{\partial V_j}{\partial Y_j} \left(\gamma_j d\ln T_j\right).$$

Using the definition of expenditure share, the welfare gains can be written as

$$d\ln V_j = \left(\frac{Y_j}{V_j}\frac{\partial V_j}{\partial Y_j}\right)\left(-\sum_{i=1}^{n} \lambda_{ij} d\ln p_{ij} + \gamma_j d\ln T_j\right). \tag{22}$$

Equation (22) is comparable to equation (12), but the derivation does not require the specific functional form in equations (10) and (11). In the absence of the three macro-level restrictions identified in ACR, equation (22) involves the additional terms $\frac{Y_j}{V_j}\frac{\partial V_j}{\partial Y_j}$ and $\gamma_j d\ln T_j$.

The first term is a money metric that changes the unit of money to utility measurement. For any homothetic preferences, the associated indirect utility function is linear in income, $V_j\left(p_{1j}, ..., p_{nj}, Y_j\right) = V_j\left(p_{1j}, ..., p_{nj}, 1\right) Y_j$. Therefore, when the assumption of homothetic preferences is made, we have $\frac{Y_j}{V_j}\frac{\partial V_j}{\partial Y_j} = 1$. The second term captures a potential change in income. ACR impose restrictions on trade balances and aggregate profits to motivate that $T_j = 0$ in their models. This paper relaxes their restrictions by allowing for $d\ln T_j \neq 0$.

From equation (19), the relationship between expenditure shares and price changes can be described as

$$\boldsymbol{DP} = \boldsymbol{\Lambda}^{-1} \cdot \boldsymbol{DL}. \tag{23}$$

Equation (23) is comparable to equation (13), but it is derived using the definition of the elasticity of substitution instead of the equilibrium condition in equation (11).

Combining equations (22) and (23) leads to a formula presented in Proposition 3.

Proposition 3. *Welfare gains from trade can be measured by*

$$dlnV_j = \left(\frac{Y_j}{V_j}\frac{\partial V_j}{\partial Y_j}\right)\left[-\left(\boldsymbol{\lambda} \cdot \boldsymbol{\Lambda}^{-1} \cdot \boldsymbol{DL}\right) + \gamma_j dlnT_j\right]. \tag{24}$$

The core concept of this formula lies in understanding how to use observables to infer changes in trade costs, which lead to welfare changes. These changes in trade costs lead to variations in prices. Although these price changes might not be directly observable, they can be deduced from the way expenditure shares shift.

Compared to the ACR formula, the generalized formula in Proposition 3 has three additional terms. First, the money metric term $\left(\frac{Y_j}{V_j}\frac{\partial V_j}{\partial Y_j}\right)$ converts the value of changes in income to welfare units. This term does not depend on the exporting country i. This money metric term is equal to one under the assumption of homothetic preferences, because welfare grows linearly in income.

Second, the term $d\ln T_j$ captures the welfare changes due to changes in income. This relaxes the ACR's macro-level restrictions on trades balances and aggregate profits. In this model, labor income is unchanged because Home's wage is the numeraire and the supply of labor is fixed. Therefore, any income change is attributed to variations in net transfers. In most trade models, these net transfers from abroad are typically absent, rendering this term zero. Section 4.3 explores scenarios where such transfers do exist. The formula in Proposition 3 shows how to generalize the ACR formula to incorporate income changes.

Third, without a specific assumption about the elasticity of substitution, adding all terms in the summation is not straightforward. It is straightforward that under constant elasticity of substitution preferences, the matrix $\boldsymbol{\Lambda}^{-1}$ is equal to $\frac{1}{1-\overline{\sigma}} I_{(n-1)}$, where $\overline{\sigma}$ is the elasticity of substitution, and $I_{(n-1)}$ is the $(n-1) \times (n-1)$ identity matrix, and equation (24) in Proposition 3 can be simplified.

Proposition 4 lays out the specific conditions under which the welfare formula in Proposition 3 can be simplified to the ACR formula.

Proposition 4. *Under constant elasticity of substitution preferences, and assuming unchanged net transfers, welfare gains from trade can be measured by*

$$dlnV_j = \frac{dln\lambda_{jj}}{1-\overline{\sigma}}, \tag{25}$$

where $\overline{\sigma}$ is the elasticity of substitution.

This formula is identical to the main result in ACR. The main point of this paper is to derive the same formula in a more general way. The new proof derives the expressions in equations (9) and (11) without the assumption that the preferences are constant-elasticity-to-scale (CES).

Equations (12) and (13) are derived using basic microeconomics theory. Compared to equation (12), equation (22) can be derived using Roy's identity. The only needed assumption is homothetic preferences, which eliminate the term related to money metric. Compared to equation (13), equation (23) can be derived using the definition of the elasticity of substitution.

Proposition 3 provides a general welfare gain from trade formula that can be applied to any trade model. To calculate the summation, it is helpful to assume that the elasticity of substitution is uniform across different goods. However, the new derivation suggests that the assumption on the elasticities can be made in the neighborhood of the initial quantities, rather than across the entire domain.

In essence, Proposition 3 points out that the welfare gains from trade formula in ACR (2012) can be applied to a larger class of models under the less restricted assumptions.

4.3 Discussions

This subsection discusses the roles of ACR's three macro-level restrictions: (A1) trade in goods is balanced; (A2) aggregate profits are a constant share of GDP; and (A3) the import demand system is CES.

4.3.1 Balanced Trade

Static trade models typically do not account for trade imbalances. The common method to include trade imbalances in these models is to treat them as fixed nominal transfers within the budget constraints of the countries involved. However, there are two major shortcomings in this approach. First, it is arguably unrealistic to assume that trade imbalances remain unaffected by changes in trade costs since trade costs influence a country's imports and exports. Second, the values of these imbalances in real terms depend on the standard monetary unit in which prices are measured and are affected by the terms of trade.

This exercise introduces trade imbalances as nominal transfers into the budget constraint. This is a common solution in quantitative exercises such as Ossa (2011) and Ossa (2014). The national income of country j in equation (15) becomes

$$Y_j = w_j L_j + TB_j,$$

where TB_j is a constant trade imbalance of country j. For the entire global economy to be balanced, the sum of all countries' trade imbalances must equal zero, i.e., $\sum_{i=1}^{n} TB_i = 0$.

Fixed trade imbalances can be compared to the phenomenon of remote work. In this comparison, a positive trade imbalance between country i and j is similar to a situation where labor from country i relocates to live in country j but continues to work remotely for their home country i. This results in a scenario where production is still occurring in country i, the workers' home country, but the consumption takes place in country j, where these workers now reside. This scenario mirrors the effects of a trade imbalance, where country j's total expenditure is larger than its total production.

In this context, the welfare change in equation (21) is modified to

$$dV_j = \sum_{i=1}^{n} \frac{\partial V_j}{\partial p_{ij}} dp_{ij} + \frac{\partial V_j}{\partial Y_j} \left(dTB_j \right). \tag{26}$$

If the trade imbalance is a fixed nominal value and does not change with trade costs, its variation is zero and welfare gains from trade do not experience an income effect. If there are changes in trade imbalances, the modified welfare gain formula is similar to the welfare gain formula presented in Proposition 3 where the net transfers are interpreted as trade imbalances. Under certain additional assumptions described in Proposition 4, the welfare gain formula would be modified to

$$d\ln V_j = \frac{d\ln \lambda_{jj}}{1 - \overline{\sigma}} + \gamma_j d\ln TB_j$$

where $\gamma_j = TB_j / Y_j$ is the share of country j's trade imbalance in its total income.

Compared to the formula in Proposition 4, there is an additional term that represents the impact of changes in trade imbalances. This modification involves two key factors: the percentage change in trade imbalances and their proportion in the total income of a country. In practice, the terms γ_j and $d\ln TB_j$ can be computed using data on a country's GDP and its current account balance.

4.3.2 Aggregate profit as a share of GDP

Consider a scenario where aggregate profits are not zero. In this case, the total income of country j in equation (15) becomes

$$Y_j = w_j L_j + \Pi_j.$$

The formula for calculating welfare gains in this situation would be:

$$dV_j = \sum_{i=1}^{n} \frac{\partial V_j}{\partial p_{ij}} dp_{ij} + \frac{\partial V_j}{\partial Y_j} \left(d\Pi_j \right). \tag{27}$$

This equation is an extension of the standard welfare change formula in equation (21). The key difference here is the term $d\Pi_j$, which reflects changes in aggregate profits.

ACR's Assumption A2 assumes that aggregate profit as a share of income is constant. This implies that aggregate profits change proportionally to income. Because labor income is assumed to be unaffected by trade costs, neither aggregate profits are affected. As a result, we have $d\ln\Pi_j = d\ln Y_j = 0$, and the ACR formula does not involve the income effect via aggregate profits.

In this paper, changes in aggregate profits can be allowed. The welfare gain formula in Proposition 4 would be modified to

$$d\ln V_j = \frac{d\ln\lambda_{jj}}{1-\overline{\sigma}} + \gamma_j d\ln\Pi_j.$$

where $\gamma_j = \Pi_j / Y_j$ is the share of country j's aggregate profits in its total income.

The modified formula has an additional term that represents a change in aggregate profits. The modification involves the percentage change of aggregate profits and the share of aggregate profits in total income.

4.3.3 Summary

ACR establish three macro-level restrictions to conclude their welfare change formula. These restrictions are sufficient to simplify the general formula in Proposition 3 to the formula in Proposition 4.

ACR's Assumptions A1 and A2 are crucial for excluding income change terms from consideration in the welfare gain formula. With a constant labor income (by choice of numeraire and fixed labor supply), any income change must stem from net transfers. The first assumption does not allow countries to have trade deficits or surpluses, which are observationally equivalent to a change in income. Meanwhile, the assumption that aggregate profit is a constant proportion of income implies that aggregate profits and income grow at the same rate. Because labor income is constant (by construction), the aggregate profit also does not change. In short, Assumptions A1 and A2 are conditions that imply $\gamma_j d\ln T_j = 0$.

Assumption A3 plays a different role in the derivation. This assumption implicitly assumes that consumer preferences are homothetic and that the elasticities of substitution across different goods are identical. The former implies $\frac{Y_j}{V_j}\frac{\partial V_j}{\partial Y_j}$ is equal to one, while the latter helps to simplify the summation.

4.4 Examples: The Melitz (2003) Model

This section illustrates how Propositions 3 and 4 are applicable to the Melitz model.

4.4.1 Setup

Consider the model in Melitz (2003). The preferences are

$$U_j = \left[\int_{\omega\in\Omega} \left(q\left(\omega\right)\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{28}$$

where $q\left(\omega\right)$ is the amount of differentiated good indexed by ω consumed by country j, and $\sigma > 1$ is the (constant) elasticity of substitution.

There is a continuum of firms with mass M. Firms pay an entry cost f_e to randomly draw productivity ϕ from the cumulative distribution function $G\left(\phi\right)$ with its associated probability density function $g\left(\phi\right)$. The wage in country i is denoted by w_i. Firms in country i face a fixed cost f_{ij} to sell their output in country j. The labor requirement in the production is $1/\phi$.

Based on the preferences, firms select a constant markup $\sigma/\left(\sigma-1\right)$ over marginal costs $w_i\tau_{ij}/\phi$. The profit of a firm with productivity ϕ in country i selling to country j is $\pi_{ij}\left(\phi\right) = p_{ij}\left(\phi\right)q_{ij}\left(\phi\right) - w_i\tau_{ij}q_{ij}\left(\phi\right)/\phi - f_{ij}$. There is a productivity threshold ϕ^*_{ij} such that $\pi_{ij}\left(\phi^*_{ij}\right) = 0$ and only firms in country i that have $\phi \geq \phi^*_{ij}$ sell to country j.

The aggregate trade flow from country i to country j is

$$X_{ij} = \int_{\phi^*_{ij}}^{\infty} p_{ij}\left(\phi\right)q_{ij}\left(\phi\right)g\left(\phi\right)d\phi. \tag{29}$$

4.4.2 Derivation

The utility function in equation (28) can be expressed in aggregate terms as

$$U_j = \left[\sum_{i=1}^{n} Q_{ij}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

where

$$Q_{ij} = \left[\int_{\phi^*_{ij}}^{\infty} \left(q\left(\phi\right)\right)^{\frac{\sigma-1}{\sigma}} g\left(\phi\right)d\phi\right]^{\frac{\sigma}{\sigma-1}}$$

is country j's composite consumption of country i's goods. This composite consumption is different from the aggregate trade flow in equation (29). Composite consumption is formulated to capture the utility derived from consuming goods from individual firms. In contrast, the aggregate trade flow is a direct summation of the total value of goods from each firm.

From the demand function, the price of composite consumption Q_{ij} is

$$P_{ij} = \left[\int_{\phi_{ij}^*}^{\infty} \left(p\left(\phi\right)\right)^{1-\sigma} g\left(\phi\right) d\phi \right]^{\frac{1}{1-\sigma}}. \tag{30}$$

The associated indirect utility function V_j is

$$V_j = U\left(P_{1j}, ..., P_{nj}, Y_j\right).$$

Using Roy's identity, the definition of expenditure share, and the assumption of homothetic preferences, the welfare gains can be written as

$$d\ln V_j = -\sum_{i=1}^{n} \lambda_{ij} d\ln P_{ij} + d\ln Y_j.$$

Because Home's wage is the numeraire and labor supply is fixed, the term $d\ln Y_j$ is equal to zero.

From the demand function and equations (29) and (30), we can show that

$$\frac{X_{ij}}{X_{jj}} = \left(\frac{P_{ij}}{P_{jj}}\right)^{1-\sigma}. \tag{31}$$

Equation (31) is useful, because it implies that we can directly measure relative changes in aggregate prices through relative changes in expenditure shares without information on the changes in productivity thresholds $d\ln\phi_{ij}^*$ or trade costs $d\ln\tau_{ij}$.

Therefore, relative changes in aggregate prices can be deduced from the observed changes in expenditure shares through

$$d\ln\lambda_{ij} - d\ln\lambda_{jj} = (1-\sigma)\left(d\ln\left(P_{ij}\right) - d\ln\left(P_{jj}\right)\right).$$

As a result, the welfare gain formula based on the Melitz model is identical to the formula in Proposition 4 that

$$d\ln V_j = \frac{d\ln\lambda_{jj}}{1-\overline{\sigma}}.$$

4.4.3 Discussion

The Melitz model introduces an additional layer of welfare gains by emphasizing the reallocation of resources from less productive to more productive firms. When trade costs decline, the Melitz model projects larger welfare gains compared to the Krugman model and the Armington model, because the Melitz model accounts for the reallocation of labor toward firms that are more efficient in production, thereby enhancing overall economic productivity.

However, when we observe the same expenditure shares, the calculated welfare gains from the

Melitz, Krugman, and Armington models align. This equivalence stems from the fact that the observed expenditure shares encapsulate the net effects of reductions in trade costs and reallocation efficiencies.

The welfare formula is remarkably versatile and capable of being applied to a wide range of trade models. This includes complicated models that incorporate global supply chains or introduce a new margin of product quality. The formula's widespread relevance is anchored in its core mechanism, which is focused on understanding how to use observables to infer price changes that eventually influence welfare.

In any trade model, regardless of its complexity or the specific mechanisms it includes, what ultimately matters is the aggregate impact on prices and, consequently, on welfare. Even in sophisticated models that incorporate global supply chains or nuances such as product quality, the formula remains applicable because it boils down to analyzing the aggregate price changes of output. These changes, which are deducted from the expenditure shares and the elasticity of substitution, provide sufficient information to measure welfare gains.

5 Conclusion

This paper revisits the welfare gain from trade formula proposed by Arkolakis, Costinot, and Rodríguez-Clare (2012). This research has ventured into unexplored dimensions, offering novel insights and expanding the applicability of existing models. This study has particularly focused on two key areas, significantly enhancing our understanding of the welfare gains from trade.

The first area extends the mechanism of the general welfare gain formula to classical trade models. Unlike the general formula for new-trade models, the extended formula for classical trade models incorporates an additional channel for welfare gains—linked to changes in income arising from alterations in the production point. This nuanced approach enables the calculation of gains from trade through observable factors such as production shares, the elasticity of output substitution, expenditure shares, and the elasticity of substitution.

The second area develops a welfare gain formula without the need for specific knowledge of consumer preferences. Starting with the standard Armington model and incorporating general preferences, I apply Roy's identity to derive key equations, leading to a general formula for welfare gains from trade. This formula introduces innovative features, including a term tied to welfare and income level, the consideration of potentially different elasticities of substitution across goods, and the possibility of trade imbalances or aggregate profits. Notably, under certain assumptions, the proposed general formula can be reduced to the well-established ACR formula.

In essence, this research has not only refined and extended existing models but has also offered a more nuanced understanding of welfare gains from trade. By broadening the scope of applicability and incorporating additional dimensions, the findings presented here lay a foundation for the further exploration and refinement of trade models, opening avenues for future research in this field.

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