Potential Buyers and Fire Sales in Financial Networks*

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Abstract

We model contagion in financial networks using an equilibrium approach. Banks in the networks are risk averse and optimize loan holdings. Contagion can occur through the interbank liabilities, and asset prices. Banks in the system are allowed to act as potential buyers buying illiquid loans from troubled banks. We find that banks with low risk aversion, the *aggressive banks*, can be helpful as good potential buyers and harmful as risk amplifiers. The system is subject to greater risk when aggressive banks hold claims between each other, and when they hold high risk loans as their prices are sensitive to shocks. When banks and loan markets are separated into non-overlapping sectors based on their areas of expertise defined by the cost of managing different types of loans, a shock in one sector is not transmitted to another sector if there are not liability linkages and the cost is sufficiently high. We use this observation to suggest a policy that separates banks during good times to limit unexpected contagion, and allows them to act as *secondary* potential buyers to save their peers during bad times, creating a self-rescue system that puts no burden on the taxpayers.

Key words– potential buyers, fire sales, contagion, illiquidity, banking network.

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1 Introduction

Asset fire sales can be a major cause of a financial crisis. When banks are hit by unexpected shocks, either directly or indirectly through contagion channels, banks can become insolvent and need to liquidate all of their assets, the major of which are illiquid loans. Banks that survive but experience losses need to sell some of their assets to pay for the losses. Because in general banks' liquid assets are low-risk assets such as cash equivalents and banks' illiquid assets are high-risk assets such as loans, risk-averse banks do not sell only liquid assets, but also part of their illiquid assets to re-optimize the risk-adjusted return of their portfolios after the shocks.[1] How much banks sell off their illiquid loans depends on the banks' attitude toward risk.[2] If there are few potential buyers who are willing to pay for the assets, the fire sale prices can be much worse (Shleifer and Vishny, 2011). The system that tends to keep some potential buyers untouched from a result of a shock can be a solution to the fire sale problem. This *self-rescue* feature avoids seeking for help from the outsiders such as the government or the central bank, and thus reduces the burden on the taxpayers.

In this paper we present an equilibrium model of financial contagion in banking networks that allows survival banks to act as potential buyers. Banks in our model are risk averse, and hence they optimize their portfolios of cash and various types of illiquid loans based on a risk-adjusted return basis. A bank with a low level of risk aversion, or an *aggressive bank*, holds a larger portion of illiquid risky loans per unit of the bank's equity. So when the equity value of the bank decreases, it reduces its loan holding more aggressively than banks that are more risk averse, or *conservative banks*. When a shock hits an aggressive bank, it can originate fire sales of illiquid loans. On the other hand, if most aggressive bank are not affected by the shock, they act as potential buyers who are willing to take a large amount of loans given a small reduction in the price, and thus help save the network from asset fire sales.

Banks in our model are also linked through interbank liabilities. If an aggressive bank, holding a large amount of loans, is hit by a large shock and becomes insolvent, the loss will be propagated to its interbank creditors who will need to sell off some of their loans, if not all. If those creditors are relatively more aggressive compared to the other banks in the network, then the sales of illiquid loans can be huge, while the non-creditor banks, who are relatively more conservative, require a deep discount in the prices in order to generate enough demand to meet the huge supply of loans. In this case, both sellers and buyers amplify the fire sale effects. On the other hand, if the creditors are the relatively more conservative banks, the loan sales will be less and the non-creditor banks, which are relatively more aggressive, would take all of the loans given just a small discount. Thus, the self-rescue ability of the network relies

[1] See Pyle (1971) for an example of banks using portfolio management to determine optimal allocation for the banks' assets.

[2] Ratti (1980) and Angelini (2000) provide evidence that banks are risk averse.

partly on how conservative and aggressive banks are linked through the interbank liabilities.

When the loan markets and banks are divided into multiple sectors, the role of banks as potential buyers can be significant during a crisis of a sector. In our model, different banks have different expertise in managing different types of loans. Managing the loan type in their expertise requires a low cost, while managing loans outside their expertise is costly. This high cost creates a barrier for banks to hold loans outside their expertise and divides the loan markets and banks into sectors. In this situation, our model shows that when a small shock hits a bank in one sector, the other banks in the sector will act as the potential buyers providing the self-rescue mechanism of the sector. However, when the sector is hit by a large shock damaging many banks in the sector, it is difficult to avoid a sharp drop in the loan price as banks do not play their role as helpful potential buyers. This can happen, for example, if the default risk of the loans in the sector jumps up markedly, affecting the equity values of all banks in the sector. If the loan price falls enough to outweight the high cost, then, and only then, banks outside the failing sector will step in and act as the potential buyers to rescue the failing sector when it is most needed.

Separating loan markets and banks into sectors helps create the *secondary potential buyers*, and thus enhances the self-rescue mechanism of the system. When banks are separated into loan sectors, a shock to one type of loans does not cause losses to banks outside the sector through the asset price channel. Thus, it keeps those outside banks safe and ready to step in to save the sector once the time comes. Each sector now acts as secondary potential buyers for the other sectors. However, interbank liability linkages may exist between banks that belong to different sectors. This channel of contagion weaken the role as the secondary potential buyers of the outside banks as losses from the failing sector can be transmitted to the banks outside the sector. Another factor that can weaken the secondary potential buyer role of the outside banks is the default correlation. Default correlations between loans from different sectors create negative hedging demands due to the substitution effect. That is, when outside banks step in to buy loans from the failing sector, they reduce the holdings of the loans in their sector as they are substitute goods, causing the price to drop. The negative hedging demand is large when the correlation is high and the fire sale loans are attractive.

This result provides an interesting policy suggestion. The regulatory body can divide loan markets into non-overlapping sectors, and require banks to choose an area of expertise (i.e. the loan sector) in which the banks are allowed to run their businesses as usual.[3] Banks running the business outside their declared area of expertise are required to pay a large amount of fee. This is to reduce the incentive of banks to create undesired contagion channel across the sectors. As a result, banks from one sector act as the secondary potential buyers for the others. The regulator should also set a limit for interbank liabilities between banks from different areas

[3]An area of expertise may consist of many types of loans. However, each area of expertise cannot overlap.

of expertise. Once a crisis is about to happen in one sector in which the fire sale can bring down the sector, the regulator may search for financially healthy sectors. If there is not any healthy sector, then it might be better to keep the fee high to avoid a cross-sector contagion and limit the losses. If there are financially healthy sectors, the regulator may choose to lower the fee for the financially healthy sectors to allow the healthy banks to step in and save the failing sector. The regulator can choose the fee level that does not cause a serious effect on the healthy sectors. Once this happens; however, the healthy sectors are contaminated by the failing sector, and the self-rescue system will not function for the next crisis. So the regulator should use this as a temporary solution and try to bring the system back to normal soon before the next crisis.

Our work is related to the literature of potential buyers during fire sales. Shleifer and Vishny (1992) develop an equilibrium model of two firms to show that the price of an asset in liquidation can fall below the value in best use because the depression causes one firm to liquidate its asset, while the other firm in the same industry, who values the asset and is the potential buyer, also has trouble raising fund during the depression period. The asset thus has to be sold to the outsider who does not know how to manage, and is willing to pay at a lower price. See also Shleifer and Vishny (2011) for the role of the firms inside the industry as the potential buyers and when they do not function well during financial crises.

Acharya and Yorulmazer (2008) consider a system of banks and outside investors in which the assets of the failed banks are auctioned to surviving banks and the outside investors. As the number of failed banks increases, the number of surviving banks decreases and the surviving banks do not have enough funds to buy all the assets, allowing the outside investors, who are inefficient users of the assets, to purchase the assets. They show that a bail out policy gives incentives for banks to herd ex-ante and in turn increases the risk of the bank failures. On the contrary, providing capital to surviving banks to acquire the failed banks' assets when the number of failed banks is large leads banks to differentiate their loan exposures. Acharya et al. (2011) extend the model and show that providing the support to surviving banks conditional on their liquid asset holdings gives incentives for banks to hold more liquid assets. In contrast to the earlier works, we propose a resolution of financial meltdown that relies on the capital inside the system by keeping them separated when the economy is sound, but allowing them to act as the secondary support when the primary potential buyers fail to rescue their sectors.

Modeling interbank liabilities linkages using network models has recently been of great interests in the contagion literature. Eisenberg and Noe (2001) provide a model of debt clearing among defaulted and non-defaulted firms, and show the existence and the condition for the uniqueness of the repayment vector. Acemoglu et al. (2015) analyze the network stability conditional on a shock, and show that when the shock is small, a more densely connected network provides better stability due to risk-sharing benefits, while when the shock is large, the dense connection creates fragility of the network by propagating shocks throughout the network. This is consistent with the experiment of Gai and Kapadia (2010) who report that financial networks are robust-but-fragile. Demange (2016) develop a threat index that accounts for firm's characteristic and the links to the other firms in the network to identify optimal intervention policies.

The literature on the financial networks of cross-liability has been extended to include the contagion channel due to asset prices. Cifuentes et al. (2005) propose a network model of interconnected financial institutions that are subject to regulatory solvency constraints. In their model, the institutions may have to sell off their illiquid assets after being hit by a shock to satisfy the constraints. As the assets are illiquid, the asset sales cause the prices to drop and further reduce the mark-to-mark values of the institutions' capital, triggering another round of asset sales. They model the price changes in response to the asset sales using an inverse demand function, and hence implicitly assume that assets are sold to institutions outside the network. Cecchetti et al. (2016) and Feinstein (2017) extend the work of Cifuentes et al. (2005) to include multiple illiquid assets and provide a proof of the existence of an equilibrium. Following the framework of Cifuentes et al. (2005), Chen et al. (2016) study the interaction between the contagion channels through the liability linkages and the asset prices. They conclude that illiquidity channel has a great potential to cause systemic-wide contagion. Greenwood et al. (2015) study the effect of fire sales in banking networks but do not consider interbank liability linkages. They assume that banks sell assets to adjust the leverage ratios back to their target levels after being hit by negative shocks. Their study shows that the systemic risk of a banking system is large if the volatile and illiquid assets are held by the most levered banks, and they suggest that illiquid assets with low volatility should be isolated from the risky ones so that they are not contaminated by those assets.

All of these works share the same assumptions embedded in the model of Cifuentes et al. (2005). That is, they assume that financial institutes sell assets purely due to the liquidity constraints or the target leverage, and the assets are sold to outsiders using an assumed inverse demand function. This differs from our model in which the financial institutes make decision on asset holdings based on a risk-adjusted returns utility, reflecting the profit-seeking and risk-averse nature of financial firms.[4] In addition, the assets in our model are sold to the institutions inside the network, and the asset prices are determined endogenously. This allows us to study a more complete picture of fire sales which are originated from the unbalance between the low demand and high supply. For a recent review of other related financial contagion literature, we refer the reader to Glasserman and Young (2016).

The rest of this paper is organized as follows: Section 2 outlines the banking network model with two channels of contagion: interbank liabilites and asset prices. Section 3 derives

[4] Aldasoro et al. (2016) consider banking networks that incorporate three channels of contagion: liquidity hoarding, interbank liabilities and fire sales. Similar to ours, they assume that banks are risk averse and maximize the expected utility of the banks' profits. However, they do not consider buyers as in our model.

the optimal asset holdings of banks. This characterizes the demands of banks which are crucial to understand the roles of banks as sellers and buyers of illiquid assets. Section 4 provides analysis of equilibrium prices before and after shocks hit the system. We conclude in Section 5.

2 Financial Network Model

2.1 Setup

Consider a three-period ($t = 0, 1, 2$) financial system with N *banks*. At time 0, each bank i holds a liquid asset or *cash* with value c_i and a portfolio of illiquid assets or *loans*. There are K types of loans which are for non-bank borrowers such as auto loans and credit card loans. These loans mature at time 2 with random payoffs, and do not pay intermediate payments. Each bank i is endowed with $\theta_{i,k}$ units of type-k loans at time 0. Banks do not create new loans after time 0. In our model, cash represents a liquid asset portion of a bank that normally provides low return with minimal risk (zero risk and zero return in our setting). On the other hand, illiquid loans represent a majority portion of the bank's asset that are typically riskier and have higher returns.

In addition to cash and loans made for non-bank borrowers, banks are endowed with shorter-term *interbank loans* between each other. These interbank loans mature at time 1. Let $l_{i,j}$ denote the claim of bank j on the asset of bank i at the maturity of the interbank loan. The interbank claims at time 1 can be summarized by matrix $L = [l_{i,j}]$ where $l_{i,i} = 0$. We assume that no new interbank loans are created after time 0.

At time 0, each bank i is also financed by deposits of d_i. We assume that the interbank liabilities and deposits are of equal seniority, and that interest rate is normalized to zero. So cash and deposits earn no interest. To summarize, the asset side of the balance sheet of each bank consists of cash, illiquid loans, and interbank loans, while the debt side of the bank consists of deposits and interbank liabilities. The equity value of the bank is equal to the asset value minus the debt value. We assume that each bank has positive equity at time 0; that is, all banks are solvent.

At time 1, the system is subject to unexpected shocks, which can be in various forms. A shock can be a *bank shock* that comes in as a surprise expense of one particular bank, reducing the bank's net worth. Such shocks could be due to frauds, litigation costs, or settlements of lawsuits.[5] A shock can also be categorized as an *asset shock* such as an increase in the default probability of one type of loans. An adverse shock to the default probability causes the loan price to drop. We model that mechanism below. After shocks are realized, banks settle their

[5]For example, the fraud in the Barings Bank caused it to collapse in 1995. The 2016 annual report of the Royal Bank of Scotland reports the loss of over 5.8 billion pounds for litigation and conduct costs.

interbank liabilities by using cash or repayments obtained from their interbank claims, or by selling their illiquid loans, or a combination of them.

Suppose that after the shocks the market price of type-k illiquid loan is $p_k, k = 1, \ldots, K$. Let $x_{i,j}$ denote the amount that bank i repays its interbank liability to bank j for $j \neq i$. If the value of the total asset of bank i is less than the value of its total debt, then the bank is insolvent and must liquidate all of its assets and distribute the proceeds to all of its creditors proportional to the face values. Otherwise, bank i repays the interbank liability in full. Let $L_i = d_i + \sum_{u \neq i}^{N} l_{i,u}$ denote the total debt of bank i. Then the amount that bank i repays to bank j is equal to

$$x_{i,j} = \frac{l_{i,j}}{L_i} \min \left\{ L_i, c_i + \sum_{k=1}^{K} p_k \theta_{i,k} + \sum_{u \neq i}^{N} x_{u,i} - v_i \right\} \tag{1}$$

We assume that each bank carries over its cash and deposits from time 0 to time 1. Given all the shocks and the market prices of illiquid loans $p = [p_1, \ldots, p_K]'$, the collection of $[x_{i,j}]$ for $i \neq j$ that satisfies (1) for each $i = 1, \ldots, N$ simultaneously is said to be an *equilibrium repayment* at the price vector p.

Let

$$e_i = \max \left\{ 0, c_i + \sum_{k=1}^{K} p_k \theta_{i,k} + \sum_{u \neq i}^{N} x_{u,i} - v_i - d_i - \sum_{u \neq i}^{N} l_{i,u} \right\} \tag{2}$$

denote the equity value of bank i. Banks that are solvent ($e_i > 0$) can now adjust their asset portfolio by buying or selling illiquid loans. We assume that each solvent bank chooses a portfolio of liquid and illiquid assets to maximize its risk-adjusted return on equity based on a mean-variance utility. Let $\hat{\theta}_{i,k}$ denote the number of units of tyep-k loan held by bank i after the adjustment, and $\tilde{R}_{i,k}(\hat{\theta}_{i,k})$ denote the values of the bank i's portfolio of the type-k illiquid loans realized at time 2. The change in the equity value from time 1 to time 2 comes from the change in the value of illiquid loans, and the cost of managing the loans. So the return on equity of bank i is

$$ROE_i = \frac{\sum_{k=1}^{K} (\tilde{R}_{i,k}(\hat{\theta}_{i,k}) - \hat{\theta}_{i,k} p_k - \hat{\theta}_{i,k} f_{i,k})}{e_i} \tag{3}$$

where $f_{i,k}$ is the cost of managing one unit of type-k loan of bank i. The cost matrix $F = [f_{i,k}]$ is used to define *expertise* of banks for different types of loans. Now the optimization problem of bank i is given by

$$[\hat{\theta}_{i,1}, \ldots, \hat{\theta}_{i,K}] = \arg\max \left\{ E[ROE_i] - \frac{\gamma_i}{2} Var(ROE_i) \right\} \tag{4}$$

subject to the budget constraint

$$\sum_{k=1}^{K} p_k \hat{\theta}_{i,k} \leq e_i + d_i \tag{5}$$

where γ_i denotes the risk-aversion parameter of bank i. Note that banks are not allowed to hold short positions on loans and hence we also need the no-short-position constraint

$$\hat{\theta}_{i,k} \geq 0, \quad k = 1, \ldots, K. \tag{6}$$

We provide the details of $\tilde{R}_{i,k}(\theta_{i,k})$ below.

We assume that illiquid loans are traded inside the financial system with N banks. So the loan prices are determined endogenously based on the market clearing condition. That is, the price vector $p = [p_1, \ldots, p_K]'$ is said to be an *equilibrium price* if the demand and supply of each loan type are equal:

$$\sum_{i=1}^{N} \theta_{i,k} = \sum_{i=1}^{N} \hat{\theta}_{i,k}, \quad k = 1, \ldots, K. \tag{7}$$

2.2 Default correlations and loan payoff distribution

There are K types of illiquid loans. At time 2, a loan of type k repays the creditor the full amount of \$1 with probability $1 - \lambda_k$ or defaults and pays nothing to the creditor with probability λ_k for $\lambda_k \in (0, 1)$. We assume that loan defaults are correlated and we model the default correlation with a Gaussian copula model. Specifically, let $M_k = \sum_{i=1}^{N} \theta_k$ denote the total number of type-k loans that are available in the system, and $\tilde{r}_{m,k}$ denote the payoff of loan m in type k, $m = 1, \ldots, M_k$. The Gaussian copula framework models default correlation through common factors. Let $\tilde{Z}_0, \tilde{Z}_1, \ldots, \tilde{Z}_K$ be independent standard normal random variables such that $\tilde{Z}_0$ represents the *market* factor, and $\tilde{Z}_k$ represents type-k factor for $k = 1, \ldots, K$. For each loan m of type k, let

$$\tilde{Y}_{m,k} = \alpha \tilde{Z}_0 + \beta_k \tilde{Z}_k + \sqrt{1 - \alpha^2 - \beta_k^2} \tilde{\epsilon}_{m,k} \tag{8}$$

where $\tilde{\epsilon}_{m,k}$'s are i.i.d. standard normal random variables for $m = 1, \ldots, M_k, k = 1, \ldots, K$ and are independent of $\tilde{Z}_0, \ldots, \tilde{Z}_K$. The parameters α and β_k are such that $\alpha \geq 0, \beta_k > 0$ and $\alpha^2 + \beta_k^2 < 1$. Observe that each $\tilde{Y}_{m,k}$ is also a standard normal random variable, and they are correlated. The correlations between $\tilde{Y}_{m,k}$'s are used to determine the default correlations between loan payoffs $\tilde{r}_{m,k}$'s based on the following relationship:

$$\tilde{r}_{m,k} = 0 \quad \text{if and only if} \quad \tilde{Y}_{m,k} \leq \Phi^{-1}(\lambda_k)$$

where $\Phi(y)$ is the cumulative distribution function of standard normal distribution at y. So loan m in type k defaults if and only if $\tilde{Y}_{m,k} \leq \Phi^{-1}(\lambda_k)$. Note that from the standard normal distribution of $\tilde{Y}_{m,k}$, the default probability is $P(\tilde{Y}_{m,k} \leq \Phi^{-1}(\lambda_k)) = \lambda_k$, as it must.

The correlations between loans depend on parameters $\alpha, \beta_1, \ldots, \beta_K$. For loans m and m' in the same type k, the correlation between $\tilde{Y}_{m,k}$ and $\tilde{Y}_{m',k}$ is $\alpha^2 + \beta_k^2$, while for loans m and m' of different types k and k', correlation between $\tilde{Y}_{m,k}$ and $\tilde{Y}_{m',k'}$ is α^2. In other words, loans of different types are less correlated than loans of the same types, and the difference is determined by β_k. With different values in $\beta_1, \ldots, \beta_K$, we can have loan types that have higher default correlation, such as corporate loans within a particular sector, and loan types that have lower default correlation, such as student loans. We have the following results:

Proposition 1 *Let $\tilde{R}_{i,k}(\theta) = \sum_{m=1}^{\theta} \tilde{r}_{m,k}$ and $\tilde{R}_{i,k'}(\theta') = \sum_{m=1}^{\theta'} \tilde{r}_{m,k'}$ denote the payoffs at time 2 of the portfolio of θ units of type-k loans, and portfolio of θ' units of type-k' loans, respectively. We have*

$$E\left[\tilde{R}_{i,k}(\theta)\right] = \theta(1-\lambda_k) \tag{9}$$

$$Var\left(\tilde{R}_{i,k}(\theta)\right) = \theta^2[\psi_k - (1-\lambda_k)^2] + \theta[1-\lambda_k-\psi_k] \tag{10}$$

$$Cov\left(\tilde{R}_{i,k}(\theta), \tilde{R}_{i,k'}(\theta')\right) = \theta\theta'[\Psi_{k,k'} - (1-\lambda_k)(1-\lambda_{k'})], \quad k \neq k' \tag{11}$$

where

$$\psi_k = \Phi_2(\Phi^{-1}(1-\lambda_k), \Phi^{-1}(1-\lambda_k); \alpha^2 + \beta_k^2)$$
$$\Psi_{k,k'} = \Phi_2(\Phi^{-1}(1-\lambda_k), \Phi^{-1}(1-\lambda_{k'}); \alpha^2)$$

and $\Phi_2(y_1, y_2; \rho)$ is the bivariate cumulative distribution function at (y_1, y_2) of standard normal random variables with correlation ρ.[6]

The expected value in (9) is simply the number of loans multiplied by the probability that a type-k loan does not default. The variance in (10) has two terms. The first term is quadratic in the number of loans, while the second term is linear. To understand this, note first that the variance of the payoff of *one unit* of a type-k loan is $\lambda_k(1-\lambda_k)$. This variance is broken into $\psi_k - (1-\lambda_k)^2$ and $1-\lambda_k-\psi_k$, the sum of which is $\lambda_k(1-\lambda_k)$. When the term $\psi_k - (1-\lambda_k)^2$ at the loan level is aggregated to the portfolio level, it gives a quadratic function of θ, while

[6]Note that we have assumed that θ is an integer, but we will rely on the same formulas (9) - (11) even when θ is real. The error from rounding the number should not change the conclusions of the consequent analyses.

the term $1 - \lambda_k - \psi_k$ gives rise to a liner function at the portfolio level. As the value of ψ_k increases from $(1 - \lambda_k)^2$ when loan defaults are independent to $1 - \lambda_k$ when they are perfectly correlated, we can view $\psi_k - (1 - \lambda_k)^2$ as a *variance-based* measure of how loans are close to perfect correlation, and view $1 - \lambda_k - \psi_k$ as a variance-based measure of how loans are close to independence. So the portfolio variance given by (10) suggests that the higher the correlation, the stronger the quadratic term, and the weaker the linear term. Finally, if loans of different types default independently ($\alpha = 0$), then $\Psi_{k,k'} = (1 - \lambda_k)(1 - \lambda_{k'})$ and thus the covariance between the values of loan portfolios in (11) is zero. It is positive if the defaults of loans of different types are positively correlated.

3 Banks' optimal portfolios

When an adverse bank shock, such as an unexpected litigation cost, hits a bank, it reduces the bank's net worth. The bank may use cash or cash equivalents to pay for the cost, reducing the liquid asset portion in the bank's balance sheet. Because liquid assets such as cash and cash equivalents are considered as risk-free or low-risk assets, this results in the bank's asset portfolio that overweights the risky loans, increasing the risk of the bank's portfolio relative to the smaller equity value. Likewise, when an adverse asset shock hits the bank, the value of the risky loans and hence the equity value reduce, making the risk profile of the bank's portfolio deviate from the optimal level. Banks are risk-averse but profit-seeking institutions. So the changes in the proportion of risk-free/risky assets relative to its equity require banks to re-adjust their asset holdings to achieve a better risk-return trade-off.

In this section we consider the banks' optimization problems and their optimal portfolios of liquid and illiquid assets. We start with the simplest case with one type of loans. Then we study the interaction between types of loans from the case of two types of loans.

3.1 One loan type

Assume that there is only one type of illiquid loans ($K = 1$). From (3) - (4) and (9) - (10), the objective of bank i is to maximize the following risk-adjusted return on equity

$$V_i(\theta) = \frac{\theta(1 - \lambda) - \theta p - \theta f_i}{e_i} - \frac{\gamma_i}{2}\left(\frac{\theta^2[\psi - (1 - \lambda)^2] + \theta[1 - \lambda - \psi]}{e_i^2}\right) \tag{12}$$

subject to the budget constraint $\theta p \leq e_i + d_i$ and no-short-position constraint $\theta \geq 0$ where we have dropped the subscript k for simplicity.

To understand the optimal number of loans held by the bank, let's suppose for the moment that the constraints are not binding, and let $\bar{\theta}_i$ denote the optimal solution for the unconstrained problem derived from the first order condition:

$$\begin{aligned}\bar{\theta}_i &= \frac{e_i}{\gamma_i}\left[\frac{(1-\lambda)-p-f_i}{\psi-(1-\lambda)^2}\right]-\frac{1}{2}\left[\frac{1-\lambda-\psi}{\psi-(1-\lambda)^2}\right] \\ &= \frac{e_i}{\gamma_i}\left[\frac{(1-\lambda)-p-f_i}{\lambda(1-\lambda)}\right]\left(\frac{\lambda(1-\lambda)}{\psi-(1-\lambda)^2}\right)-\frac{1}{2}\left[\frac{1-\lambda-\psi}{\psi-(1-\lambda)^2}\right]. \end{aligned} \tag{13}$$

Observe that $\bar{\theta}_i$ has two components. The first component has the mean-variance spirit. It suggests that the bank should hold more loans if it has large equity value (large e_i), low risk aversion (small γ_i), or if each loan has high expected profit after cost (high $(1-\lambda)-p-f_i$), and low variance (low $\lambda(1-\lambda)$). This mean-variance term is scaled up by a factor of $\lambda(1-\lambda)/[\psi-(1-\lambda)^2] \geq 1$. Given a fixed default probability λ, this factor increases as the default correlation decreases (smaller $\psi-(1-\lambda)^2$). This is intuitive as a lower correlation provides better risk-return trade-off and hence increases the demand for loans. That is, banks should hold more loans if each loan provides a good risk-adjusted return (mean-variance term), and the portfolio has low risk as loan defaults are less correlated (scaling factor term).

This first component of the optimal loan holding comes from two effects. The first effect is the *wealth effect* through the equity value e_i. The more equity the bank has, the larger the loan demand. The second effect is the *price effect* in the expected profit. When the price increases, the expected return decreases, reducing the loan demand. What is interesting is that banks also hold loans. So as the price increases, the equity values of banks increase and create more demands for loans due to the wealth effect. On the other hand, the price effect reduces the demands. The combination of these two effects from all banks in the system determines the final price at an equilibrium, as discussed in Section 4.

The second component of the optimal holding is a downward adjustment. Its value depends on how correlated loan defaults are. The lower the correlation, the higher the size of the adjustment. When the equity values of banks are large (e.g. millions or billions of dollars), the effect of the second component is minimal. However, when banks are close to insolvency, the second component can be relatively significant.

Now let us bring back the budget and no-short-position constraints. Because the objective function (4) is quadratic in θ, it is easy to obtain the optimal solution with the constraints from the solution of the unconstrained problem. The following proposition gives the result.

Proposition 2 *The optimal loan holding for bank i when there is one type of loans is*

$$\hat{\theta}_i = \begin{cases} 0 & \text{if } \bar{\theta}_i \leq 0 \\ \bar{\theta}_i & \text{if } 0 < \bar{\theta}_i < (e_i+d_i)/p \\ (e_i+d_i)/p & \text{if } \bar{\theta}_i \geq (e_i+d_i)/p \end{cases}. \tag{14}$$

So when the bank is close to insolvency (the second downward adjustment component dominates), or when it has higher cost of managing loans ($f_i > 1 - \lambda - p$), it is optimal for the bank not to hold any loans. In contrast, when loans are very attractive, the number of loans held is capped by the budget constraint. This implies that the bank does not hold cash. However, such a situation rarely occurs because the equity decreases when the price decreases as the bank holds loans. With a large leverage ratio in the banking industry ($d_i \gg e_i$), the constraint is unlikely to be binding.

3.2 Two loan types

Assume that there are two types of loans ($K = 2$). The bank i's problem is to maximize the following objective function

$$\begin{aligned} V_i(\theta_1, \theta_2) = & \sum_{k=1}^{2} \left(\frac{\theta_k(1-\lambda_k) - \theta_k p_k - \theta_k f_{i,k}}{e_i} \right) \\ & - \frac{\gamma_i}{2} \sum_{k=1}^{K} \left(\frac{\theta_k^2[\psi_k - (1-\lambda_k)^2] + \theta_k[1-\lambda_k-\psi_k]}{e_i^2} \right) \\ & - \gamma_i \left(\frac{\theta_1\theta_2[\Psi_{1,2} - (1-\lambda_1)(1-\lambda_2)]}{e_i^2} \right) \end{aligned}$$

subject to the budget constraint $\theta_1 p_1 + \theta_2 p_2 \leq e_i + d_i$ and no-short-position constraint $\theta_k \geq 0, k = 1, 2$. Again, we first consider the case without constraints. The first order condition yields the following optimal solution for the unconstrained problem:

$$\begin{bmatrix} \theta^*_{i,1} \\ \theta^*_{i,2} \end{bmatrix} = \frac{1}{1 - \eta_{1,2}[\Psi_{1,2} - (1-\lambda_1)(1-\lambda_2)]} \begin{bmatrix} \bar{\theta}_{i,1} - \eta_{1,2}(\psi_2 - (1-\lambda_2)^2)\bar{\theta}_{i,2} \\ \bar{\theta}_{i,2} - \eta_{1,2}(\psi_1 - (1-\lambda_1)^2)\bar{\theta}_{i,1} \end{bmatrix} \quad (15)$$

where

$$\eta_{1,2} = \frac{\Psi_{1,2} - (1-\lambda_1)(1-\lambda_2)}{(\psi_1 - (1-\lambda_1)^2)(\psi_2 - (1-\lambda_2)^2)}$$

and $\bar{\theta}_{i,k}$ is $\bar{\theta}_i$ given by (13) in the one-type case with subscript k for the type-k loans. The term $\eta_{1,2}$ contains the correlation between defaults of type-1 loans and type-2 loans. Its value is zero when type-1 loans and type-2 loans are independent ($\alpha = 0$) and goes up as the correlation goes up (but keeping $\alpha^2 + \beta_k^2$ constant). To see the interaction between the two types of loans on the loan demand, let us focus on the demand of type-1 loans, or $\theta^*_{i,1}$. There are two components in $\theta^*_{i,1}$. The first one is the optimal holding from the one-type case $\bar{\theta}_{i,1}$ and the second component represents the hedging demand between the two types of loans. Suppose that $\bar{\theta}_{i,1}$, $\bar{\theta}_{i,2}$ and $\theta^*_{i,1}$ are positive. If type-1 and type-2 loans are independent, the second term disappears as $\eta_{1,2} = 0$, and in this case $\theta^*_{i,k} = \bar{\theta}_{i,k}$. When the correlation is positive, the second component creates a negative hedging demand on type-1 loans. The hedging demand depends on how attractive the type-2 loans are. The more attractive they are (large $\bar{\theta}_{i,2}$), the more negative the hedging demand, given everything else constant. That is, there is a *substitution effect* between the two loan types if the correlation is positive.

This substitution effect plays an important role on the cross-asset contagion channel. Suppose that a fire sale of type-1 loans from one bank reduces the price of the loans, making them more attractive (price effect), which in turn reduces the demand of type-2 loans of other banks due to the larger (more negative) hedging demand (substitution effect). This may trigger a fire sale on type-2 loans, and hence a contagion across the two loan types. The outcome can be much worse as there are interbank liability and common holding of type-1 loans channels that can transmit losses to other banks, following the fire sales of type-1 and type-2 loans. Lower equity values further reduces the loan demands (wealth effect) and reinforce the asset fire sales. We discuss the effect of contagion through different channels and their interaction in the subsequent sections.

To finish this section, we consider the cases when one or both of the no-short-position constraints are binding. Similar to the case of one type of loan, the budget constraint is unlikely to be binding at any equilibrium given sufficiently large leverage ratios. With the Lagrange multiplier technique, we obtain the following optimal holdings of the portfolio of loans:

Proposition 3 *Suppose the budget constraint is not binding for the case of two loan types. Then the optimal loan holdings for bank i is*

$$(\hat{\theta}_{i,1}, \hat{\theta}_{i,2}) = \begin{cases} (\theta^*_{i,1}, \theta^*_{i,2}) & \textit{if } \theta^*_{i,1} > 0 \textit{ and } \theta^*_{i,2} > 0 \\ (\bar{\theta}_{i,1},\ 0\ ) & \textit{if } \theta^*_{i,1} > 0 \textit{ and } \theta^*_{i,2} \leq 0 \\ (\ 0\ , \bar{\theta}_{i,2}) & \textit{if } \theta^*_{i,1} \leq 0 \textit{ and } \theta^*_{i,2} > 0 \\ (\ 0\ ,\ 0\ ) & \textit{if } \theta^*_{i,1} \leq 0 \textit{ and } \theta^*_{i,2} \leq 0 \end{cases}. \tag{16}$$

We now show that the negative demand always reduces the optimal holdings. Consider the case when $\theta^*_{i,1}$ is positive. From (15), it must be that

$$\bar{\theta}_{i,1} > \eta_{1,2}(\psi_2 - (1-\lambda_2)^2)\bar{\theta}_{i,2}.$$

Hence, for $\eta_{1,2} > 0$, we have

$$\begin{aligned} \theta^*_{i,2} &= \frac{\bar{\theta}_{i,2} - \eta_{1,2}(\psi_1 - (1-\lambda_1)^2)\bar{\theta}_{i,1}}{1 - \eta_{1,2}[\Psi_{1,2} - (1-\lambda_1)(1-\lambda_2)]} \\ &< \frac{\bar{\theta}_{i,2} - \eta_{1,2}(\psi_1 - (1-\lambda_1)^2)\left(\eta_{1,2}(\psi_2 - (1-\lambda_2)^2)\bar{\theta}_{i,2}\right)}{1 - \eta_{1,2}[\Psi_{1,2} - (1-\lambda_1)(1-\lambda_2)]} \\ &= \bar{\theta}_{i,2}. \end{aligned}$$

Similarly, if $\theta^*_{i,2} > 0$, we have $\bar{\theta}_{i,1} > \theta^*_{i,1}$. Thus, when $\theta^*_{i,1}$ and $\theta^*_{i,2}$ are both positive, the optimal holding of the type-k loans is $\hat{\theta}_{i,k} = \theta^*_{i,k} \geq \bar{\theta}_{i,k}$ where the inequality holds if, and only if, the loans of different types have zero default correlation, or $\eta_{1,2} = 0$. Thus, the hedging demand always reduces the optimal loan holdings.

4 Equilibrium Analysis

Prior literature on fire sales in financial networks typically assumes an inverse demand function characterizing the price change as a function of the aggregate sales. This implicitly suggests that fire sale assets are sold to buyers outside the financial system. This contradicts to the fact that a large portion of the bank assets are loans which are costly to manage by non-bankers. So it is wiser for banks to sell their assets to other banks that are more efficient buyers who are willing to pay higher prices for the assets. This situation is reasonable particularly when a shock hits one bank, and the remaining banks have sufficient funds to buy the troubled bank's assets. When there is an adverse asset shock to one type of loans, demands for the loans from banks holding that type of loans will decrease due to the wealth effect, but other banks that do not hold that loan type can be potential buyers, willing to pay for the loans, and hence help reduce the effect of the fire sale. We study the effect of potential buyers in the banking networks by allowing the prices of illiquid loans to be determined endogenously. We first provide the formal definition of our equilibrium.

Definition 1 *Given the banks' balance sheets* $(c = [c_i], \theta = [\theta_{i,k}], d = [d_i], L = [l_{i,j}])$*, banks' characteristics* $(\gamma = [\gamma_i], F = [f_{i,k}])$*, and the illiquid loans default distribution* $(\lambda = [\lambda_k], \alpha, \beta = [\beta_k])$*, an equilibrium triplet of repayments, holdings, and prices* $(X = [x_{i,j}], \hat{\theta} = [\hat{\theta}_{i,k}], p = [p_k])$ *at time* 1 *is such that*

1. *Repayment equilibrium: Equations (1)-(2) hold for all banks* $i = 1, \ldots, N$*,*
2. *Bank optimization: Each bank* i *maximizes mean-variance utility (4) subject to constraints (5)-(6),* $i = 1, \ldots, N$*.*
3. *Market clearing: Equation (7) holds.*

To study the effect of shocks and roles of banks on transmitting and absorbing shocks, we assume that without shocks to the system, there is an equilibrium in which all the interbanks liabilities are fully repaid at time 1. That is, without shocks, all banks are solvent ($x_{i,j} = l_{i,j}$). In other words, there is a price vector $p = [p_k]$ and a loan holding matrix $\hat{\theta} = [\hat{\theta}_{i,k}]$ such that the triplet $(L, \hat{\theta}, p)$ is an equilibrium triplet for the network at time 1.

4.1 One loan type

This section considers a network with one type of loans ($K = 1$), and hence we drop subscript k that refers to the loan type. We consider three scenarios: before shocks, after a bank shock, and after an asset shock.

4.1.1 Before shocks

Consider the case of no shocks with an equilibrium triplet $(L, \hat{\theta}, p)$ for which all banks are solvent. Let

$$\bar{c}_i = c_i + \sum_{j \neq i}^{N} l_{j,i} - d_i - \sum_{j \neq i}^{N} l_{i,j}$$

denote the excess cash position over the deposit after the repayment settlement of bank i. This value is typically negative as cash c_i is much smaller than deposit d_i, and interbank liabilities $l_{i,j}$'s are relatively small. The equity value of bank i is the sum of the value of the loan portfolio and its excess cash:

$$e_i = \theta_i p + \bar{c}_i > 0. \tag{17}$$

We assume further that none of the banks is close to be insolvent ($e_i \gg 0$), and that all banks are equally good at managing loans and thus have the same managing cost $f_i \equiv f < 1 - \lambda - p$.[7] So it follows from (13) that $\hat{\theta}_i = \bar{\theta}_i > 0, i = 1, \ldots, N$. Now consider the total demand of loans in the system. From (13) and (17), we have the total loan demand is

$$\begin{aligned}
\Theta_D &= \sum_{i=1}^{N} \left\{ \frac{e_i}{\gamma_i} \left[\frac{(1-\lambda) - p - f}{\psi - (1-\lambda)^2} \right] - \frac{1}{2} \left[\frac{1 - \lambda - \psi}{\psi - (1-\lambda)^2} \right] \right\} \\
&= \sum_{i=1}^{N} \left\{ \left(\frac{\theta_i p + \bar{c}_i}{\gamma_i} \right) \left[\frac{(1-\lambda) - p - f}{\psi - (1-\lambda)^2} \right] - \frac{1}{2} \left[\frac{1 - \lambda - \psi}{\psi - (1-\lambda)^2} \right] \right\} \\
&= - \left(\sum_{i=1}^{N} \frac{\theta_i}{\gamma_i} \right) \frac{p^2}{\psi - (1-\lambda)^2} + \left(\sum_{i=1}^{N} \frac{\theta_i}{\gamma_i} (1 - \lambda - f) - \sum_{i=1}^{N} \frac{\bar{c}_i}{\gamma_i} \right) \frac{p}{\psi - (1-\lambda)^2} \\
&\quad + \left(\sum_{i=1}^{N} \frac{\bar{c}_i}{\gamma_i} \right) \left[\frac{1 - \lambda - f}{\psi - (1-\lambda)^2} \right] - \frac{N}{2} \left[\frac{1 - \lambda - \psi}{\psi - (1-\lambda)^2} \right].
\end{aligned} \tag{18}$$

As we can see, the total demand in (18) is a concave quadratic function of p. This is due to

[7] We use the differences in the managing costs in the case of multiple loan types to consider the effect of banks' expertise on the stability of the network.

the combination of the wealth effect and the price effect. It is easy to see that

$$\bar{p} = \frac{1}{2}(1 - \lambda - f - \zeta)$$

is the price at which the price effect and the wealth effect of the total demand are equal ($\partial\Theta_D/\partial p = 0$) where

$$\zeta = \frac{\sum_{i=1}^{N} \bar{c}_i/\gamma_i}{\sum_{i=1}^{N} \theta_i/\gamma_i}. \tag{19}$$

When the price is lower than $\bar{p}$, an increase in the price increases the loan demand as the wealth effect dominates the price effect. When the price is higher than $\bar{p}$, an increase in the price lowers the loan demand as the price effect dominates the wealth effect. This demand behavior applies to each individual bank's loan demand in (13) as well. Precisely, the price at which both price and wealth effects equal for the demand of bank i is

$$\bar{p}_i = \frac{1}{2}\left(1 - \lambda - f - \frac{\bar{c}_i}{\theta_i}\right).$$

Let

$$\Theta_S = \sum_{i=1}^{N} \theta_i$$

denote the total number of units of loans available in the system. This represents the total supply of loans. The market clearing condition (7) dictates that the demand and supply are equal at each equilibrium: $\Theta_D = \Theta_S$. Since the total supply is fixed and the total demand is a concave quadratic function of p, we have the following result:

Theorem 1 *Suppose none of the banks is insolvent and the repayment matrix is $X = L$ in the case of one loan type. Then an equilibrium price exists if, and only if,*

$$(1 - \lambda - f + \zeta)^2 \geq 4\left[\frac{(\psi - (1-\lambda)^2)\Theta_S + \frac{N}{2}(1 - \lambda - \psi)}{\sum_{i=1}^{N} \theta_i/\gamma_i}\right], \tag{20}$$

and in that case the equilibrium prices are given by

$$p = \frac{1}{2}(1 - \lambda - f - \zeta) \pm \frac{1}{2}\left\{(1 - \lambda - f + \zeta)^2 - 4\left[\frac{(\psi - (1-\lambda)^2)\Theta_S + \frac{N}{2}(1 - \lambda - \psi)}{\sum_{i=1}^{N} \theta_i/\gamma_i}\right]\right\}^{1/2}. \tag{21}$$

The equilibrium price is unique if, and only if, the inequality in (20) is binding and the equilibrium price is $p = (1 - \lambda - f - \zeta)/2$.

To further investigate the equilibrium prices given by (21), we apply the first-order Taylor approximation for the function of the form $f(x) = (a^2 - x)^{1/2}$ around the point $x = 0$ to the last term in (21): $f(x) \approx |a| - x/2|a|$ where a is a constant. From the inequality in (17) and (19), it is clear that $\zeta > -p$. From the assumption that $\bar{\theta}_i > 0$, (13) implies that $1 - \lambda - f - p > 0$. So we have $1 - \lambda - f + \zeta > 0$. Thus, the Taylor approximation gives the following two equilibrium prices:

$$p^h \approx 1 - \lambda - f - \frac{(\psi - (1-\lambda)^2)\Theta_S + \frac{N}{2}(1 - \lambda - \psi)}{\sum_{i=1}^{N} \frac{\theta_i}{\gamma_i}(1 - \lambda - f) + \sum_{i=1}^{N} \frac{\bar{c}_i}{\gamma_i}} \tag{22}$$

$$p^l \approx -\frac{\sum_{i=1}^{N} \frac{\bar{c}_i}{\gamma_i}}{\sum_{i=1}^{N} \frac{\theta_i}{\gamma_i}} + \frac{(\psi - (1-\lambda)^2)\Theta_S + \frac{N}{2}(1 - \lambda - \psi)}{\sum_{i=1}^{N} \frac{\theta_i}{\gamma_i}(1 - \lambda - f) + \sum_{i=1}^{N} \frac{\bar{c}_i}{\gamma_i}}. \tag{23}$$

Note that $p^h > \bar{p} > p^l > 0$.

As it will become clear later that an asset fire sale may occur at the equilibrium price p^h, but not at p^l, we will focus on p^h. From (22), the equilibrium price p^h is equal to the expected payoff after cost, or $1 - \lambda - f$, minus the *premium*. The premium is the expected profit required by the banks for holding the risky loans. This premium depends on the *riskiness* of the loan

$$U = (\psi - (1-\lambda)^2)\Theta_S + \frac{N}{2}(1 - \lambda - \psi). \tag{24}$$

This riskiness U is the combination of the variance component measuring the closeness to the perfect correlation $\phi - (1 - \lambda)^2$ and the variance component measuring the closeness to the independence $1 - \lambda - \psi$. When the number of loans available in the system is large, the first term is important. On the other hand, the second becomes large when the number of banks in the system is large. The risk premium is high when the loan has high level of riskiness.

The premium also depends on the *risk-aversion-adjusted wealth* of the banking system. To see this recall that equity of bank i is

$$e_i = \theta_i p + \bar{c}_i.$$

As p^h is an approximated price, we replace p by its expected payoff after cost, which is $1 - \lambda - f$. This gives

$$e_i \approx \theta_i(1 - \lambda - f) + \bar{c}_i.$$

Each equity value is scaled by the bank's risk aversion parameter γ_i as the one unit of equity of a more risk-averse bank (high γ) is worth less than that of a less risk-averse bank (low γ) in terms of the loan demand (see (13)). Then we sum over all banks to get the risk-aversion-adjusted wealth

$$\sum_{i=1}^{N} \frac{e_i}{\gamma_i} \approx \sum_{i=1}^{N} \frac{\theta_i}{\gamma_i}(1-\lambda-f) + \sum_{i=1}^{N} \frac{\bar{c}_i}{\gamma_i}.$$

When the risk-aversion-adjusted wealth of the system is high, banks have more cash to pay for the loans, pushing the price up, and thus a lower premium. In the subsequent sections, we can explain changes in the equilibrium price p^h based on the changes in the expected payoff and/or the premium (risk and wealth).

4.1.2 After a bank shock

Let us first focus on the equilibrium price p^h, and study the change in the equilibrium price p^h in response to a bank shock. As mentioned earlier, $p^h > \bar{p}$ so the price effect of the aggregate demand is stronger than the wealth effect. If $\bar{c}_i/\theta_i$ is close to ζ, this is true for the individual bank's demand too. For the discussion below, we assume that the price effect is stronger than the wealth effect for each individual bank's demand at the equilibrium price p^h.

Suppose there is an adverse bank shock of size v_j hitting bank j. Let $\bar{\theta}_j(v_j)$ denote the value of $\bar{\theta}_j$ after an adverse shock of size v_j on bank j. This notation is used similarly for the other variables. If v_j is sufficiently small so that bank j still has positive holding in the loans, or $\bar{\theta}_j(v_j) > 0$, then the value of the excess cash of bank j after the shock is $\bar{c}_j(v_j) = \bar{c}_j - v_j$. Thus the new equilibrium price is

$$p^h(v_j) \approx 1-\lambda-f - \frac{(\psi-(1-\lambda)^2)\Theta_S + \frac{N}{2}(1-\lambda-\psi)}{\sum_{i=1}^{N} \frac{\theta_i}{\gamma_i}(1-\lambda-f) + \sum_{i=1}^{N} \frac{\bar{c}_i}{\gamma_i} - \frac{v_j}{\gamma_j}}. \tag{25}$$

So after a small bank shock v_j, the equilibrium price becomes lower due to a lower risk-aversion-adjusted wealth. What is interesting is that the effect of a shock of the same size on the equilibrium price depends on the risk-aversion parameter of the bank being hit. A small shock hitting a *conservative* bank (high γ) yields a smaller impact on the equilibrium price than a shock of the same size hitting an *aggressive* bank (low γ). This is due to the lower sensitivity of the loan demand to a one-unit decrease in the equity value, which can be seen in (13). Since everything remains the same for all other banks $j' \neq j$ except all banks see the new equilibrium price, each bank adjusts its loan holding based purely on the price change. As the price effect dominates the wealth effect at p^h, all of the other banks $j' \neq j$ act as the potential buyers and increase their loan holdings in response to the lower price. Thus, bank j

has to hold fewer loans at the new equilibrium. Note that without the shock, the price effect of bank j is stronger than the wealth effect. But since the external shock v_j reduces the equity value in addition to the effect from the lower equilibrium price, it results in a loan sell-off for bank j.

Let us consider a larger shock. Suppose that v_j is large enough to make $e_j(v_j) < 0$, but not enough to make the other banks insolvent. So bank j sells all of its loans at a fire sale price. In addition, it spreads the loss to its *neighbor* banks (the banks that hold interbank claims on the assets of bank j) through the interbank liability linkages. The loss to bank $j' \neq j$ due to the direct interbank liability with bank j is

$$l_{j,j'} \min\left\{\frac{v_j - (\theta_j p^h(v_j) + \bar{c}_j)}{L_j}, 1\right\}.$$

The new equilibrium price is

$$\begin{aligned} p^h(v_j) \approx\ & 1 - \lambda - f \\ & - \frac{(\psi - (1-\lambda)^2)\Theta_S + \frac{N}{2}(1 - \lambda - \psi)}{\sum_{i\neq j}^{N} \frac{\theta_i}{\gamma_i}(1 - \lambda - f) + \sum_{i\neq j}^{N} \frac{\bar{c}_i}{\gamma_i} - \sum_{i\neq j}^{N} \frac{l_{j,i}}{\gamma_i} \min\left\{\frac{v_j - (\theta_j p^h(v_j) + \bar{c}_j)}{L_j}, 1\right\}}. \end{aligned} \tag{26}$$

The contagion through the interbank liability channel reduces the equity values of neighbor banks that hold claims on the asset of bank j. The reduction in the equity values lowers their demands for loans, and causes the price to drop further. The impact on the price depends on the neighbor bank j''s ratio $l_{j,j'}/\gamma_{j'}$. The impact is large if the neighbor bank has a large claim on the asset of bank j and it is an aggressive bank (small γ). This suggests that interbank liabilities between aggressive banks amplify the fire sale effect in the network.

Now consider p^l. When there is a small bank shock of size v_j on bank j, the new equilibrium price is

$$p^l(v_j) \approx -\frac{\sum_{i=1}^{N} \frac{\bar{c}_i}{\gamma_i} - \frac{v_j}{\gamma_j}}{\sum_{i=1}^{N} \frac{\theta_i}{\gamma_i}} + \frac{(\psi - (1-\lambda)^2)\Theta_S + \frac{N}{2}(1 - \lambda - \psi)}{\sum_{i=1}^{N} \frac{\theta_i}{\gamma_i}(1 - \lambda - f) + \sum_{i=1}^{N} \frac{\bar{c}_i}{\gamma_i} - \frac{v_j}{\gamma_j}}$$

which is higher than the equilibrium price before the shock, or p^l, where, similar to the case of p^h, we have assumed that all of the banks are solvent after this small shock. Assuming that at p^l the wealth effect dominates the price effect for all banks, we have the increase in the price results in higher demands for loans for all other banks $j' \neq j$. As we can see this equilibrium does not correspond to a fire sale as the selling price increases after a shock arrives. In addition, banks should agree to choose the equilibrium with the higher price to maximize their net worth. So we will focus on p^h from now on.

4.1.3 After an asset shock

This section considers the effect of an asset shock on the equilibrium price. Suppose the default probability of each loan increases from λ to λ' and none of the banks are insolvent after the shock. It can be shown that the terms $\psi - (1-\lambda)^2$ and $1 - \lambda - \psi$ are increasing in λ for $\lambda \in (0, 0.5)$ but are decreasing in λ for $\lambda \in (0.5, 1)$.[8] Since the typical values of λ are less than 0.5, p^h decreases as the default probability increases (see (22)). When the price effect is stronger than the wealth effect as we assume here, each bank should tend to increase its holdings in the loans following the shock. However, as the expected value declines and the risk rises, the worsen loan characteristic reduces the loan demands and this brings the price to the new equilibrium. When the price drops, so do banks' equity values. Once the default probability is large enough, it may trigger a default of a bank, and the losses are transmitted through the interbank liability linkages, further reducing the equity of other banks. This reinforces the fire sale in the network.

Let $p^*(i)$ denote the price at which bank i's equity reaches zero due to the increase in the default probability. We have

$$p^*(i) = p^h - \frac{e_i}{\theta_i}$$

where p^h and e_i are the equilibrium price and the bank's equity before the shock, respectively. Let us call $p^*(i)$ the *critical price* of bank i. When the default probability rises, the bank that has the highest critical price can be insolvent first. As we can see, that *critical bank* is the bank that initially holds the largest number of loans per one unit of its equity value. If all banks initially hold the loans at the optimal holding level as suggested by (13), the most aggressive bank with the lowest risk aversion parameter tends to be insolvent first when the default probability increases. If the critical bank has large liabilities with other aggressive banks in the network, the contagion effect is much larger once it becomes insolvent.

4.2 Two loan types with equal costs

We now consider the case with two types of loans ($K = 2$). Here we assume that all banks have the same level of expertise in managing loans, and hence the same managing costs. That is, we assume that $f_{i,k} \equiv f_k < 1 - \lambda_k$, for $k = 1, 2$ and $i = 1, \ldots, N$. Again we consider the equilibrium prices before shocks, after a bank shock, and after an asset shock. The main difference between the one-type and two-type cases we consider here is that when there are

[8] This can be seen from the fact that $\frac{\partial \psi}{\partial \lambda} = -2\Phi\left(\Phi^{-1}(1-\lambda)\sqrt{\frac{1-\rho}{1+\rho}}\right)$ where $\rho = \alpha^2 + \beta^2$.

two types of loans, the demands of the loans of different types interact through the hedging demand and the wealth effect, causing the cross-asset contagion.

4.2.1 Before shocks

Let $(L, \hat{\theta}, p)$ denote the equilibrium triplet at time 1 before shocks, and assume that at the equilibrium all banks hold both types of loans, or $\hat{\theta}_{i,k} = \theta^*_{i,k} > 0$ for all $i = 1, \ldots, N$ and $k = 1, 2$. From (15), this implies that $\bar{\theta}_{i,k} > 0$ for all i and k. Using the optimal holding condition (15) and the clearing condition (7), it can be shown that the equilibrium price vector $p = [p_1, p_2]'$ satisfies the following system of equations:

$$U_1 = (1 - \lambda_1 - f_1 - p_1)\left[\left(\sum_{i=1}^{N} \frac{\theta_{i,1}}{\gamma_i}\right) p_1 + \left(\sum_{i=1}^{N} \frac{\theta_{i,2}}{\gamma_i}\right) p_2 + \sum_{i=1}^{N} \frac{\bar{c}_i}{\gamma_i}\right] \tag{27}$$

$$U_2 = (1 - \lambda_2 - f_2 - p_2)\left[\left(\sum_{i=1}^{N} \frac{\theta_{i,1}}{\gamma_i}\right) p_1 + \left(\sum_{i=1}^{N} \frac{\theta_{i,2}}{\gamma_i}\right) p_2 + \sum_{i=1}^{N} \frac{\bar{c}_i}{\gamma_i}\right] \tag{28}$$

where

$$U_1 = (\psi_1 - (1-\lambda_1)^2)\Theta_{S,1} + \frac{N}{2}(1 - \lambda_1 - \psi_1) + (\Psi_{1,2} - (1-\lambda_1)(1-\lambda_2))\Theta_{S,2}$$

$$U_2 = (\psi_2 - (1-\lambda_2)^2)\Theta_{S,2} + \frac{N}{2}(1 - \lambda_2 - \psi_2) + (\Psi_{1,2} - (1-\lambda_1)(1-\lambda_2))\Theta_{S,1}$$

and $\Theta_{S,k} = \sum_{i=1}^{N} \theta_{i,k}$ denotes the total number of type-k loans available in the system. The quantity U_k captures the risk of type-k loans similar to U given by (24) for the one-type case. The difference is that U_k contains an additional term due to the risk from the hedging demand, which is zero if the default correlation between the two types of loans is zero.

Observe from (27) - (28) that the loan prices at the equilibrium have linear relationship:

$$\frac{1 - \lambda_1 - f_1 - p_1}{U_1} = \frac{1 - \lambda_2 - f_2 - p_2}{U_2}. \tag{29}$$

This relationship suggests that the expected profits per unit risk of the two loan types are equal at each equilibrium. Solving (27) - (28), we obtain the following results:

Theorem 2 *Suppose none of the banks is insolvent and the repayment matrix is $X = L$ in the case of two loan types with $f_{i,k} = f_k$ for all $i = 1, \ldots, N, k = 1, 2$. Then an equilibrium price vector exists if, and only if,*

$$(1 - \lambda_k - f_k + \zeta_k)^2 \geq \frac{4U_k^2}{\left(\sum_{i=1}^{N} \theta_{i,k}/\gamma_i\right) U_k + \left(\sum_{i=1}^{N} \theta_{i,k'}/\gamma_i\right) U_{k'}}, \tag{30}$$

for $k, k' = 1, 2$ and $k \neq k'$, and in that case the equilibrium price vectors $p = [p_1, p_2]'$ are given by

$$p_1 = \frac{1}{2}(1-\lambda_1-f_1-\zeta_1) \pm \frac{1}{2}\left\{(1-\lambda_1-f_1+\zeta_1)^2 - \frac{4U_1^2}{\left(\sum_{i=1}^N \theta_{i,1}/\gamma_i\right)U_1 + \left(\sum_{i=1}^N \theta_{i,2}/\gamma_i\right)U_2}\right\}^{1/2} \tag{31}$$

where

$$\zeta_1 = \frac{\sum_{i=1}^N \frac{\theta_{i,2}}{\gamma_i}\left[U_1(1-\lambda_2-f_2) - U_2(1-\lambda_1-f_1)\right] + \sum_{i=1}^N \frac{\bar{c}_i}{\gamma_i}}{\sum_{i=1}^N \frac{\theta_{i,1}}{\gamma_i}U_1 + \sum_{i=1}^N \frac{\theta_{i,2}}{\gamma_i}U_2} \tag{32}$$

and the corresponding p_2 can be determined from the linear relationship (29).

As mentioned, we focus on the price vector at which the fire sales may occur after a shock. Using the first-order Taylor approximation as in the one-type case, the interested equilibrium price vector $p^h = [p_1^h, p_2^h]'$ is given by

$$p_k^h \approx 1-\lambda_k-f_k - \frac{U_k}{\sum_{i=1}^N \frac{\theta_{i,1}}{\gamma_i}(1-\lambda_1-f_1) + \sum_{i=1}^N \frac{\theta_{i,2}}{\gamma_i}(1-\lambda_2-f_2) + \sum_{i=1}^N \frac{\bar{c}_i}{\gamma_i}} \tag{33}$$

for $k = 1, 2$. This is similar to p^h of the one-type case given by (22). However, the denominator of the premium term now contains the expected payoff after cost of both types of loans. In addition, the numerator in the premium term, or U_k, has an extra hedging demand component which links the default probability of one type of loans to the price of the other type of loans. We discuss the implications below.

4.2.2 After a bank shock

Consider a small adverse bank shock on bank j of size v_j. Assume that this shock does not cause any insolvency in the banking system. The equilibrium price changes from $p^h = [p_1^h, p_2^h]'$ as given by (33) to

$$p_k^h(v_j) \approx 1-\lambda_k-f_k - \frac{U_k}{\sum_{i=1}^N \frac{\theta_{i,1}}{\gamma_i}(1-\lambda_1-f_1) + \sum_{i=1}^N \frac{\theta_{i,2}}{\gamma_i}(1-\lambda_2-f_2) + \sum_{i=1}^N \frac{\bar{c}_i}{\gamma_i} - \frac{v_j}{\gamma_j}}.$$

As we can see, the effect is similar to the one-type case; that is, the prices of both types of loans reduce due to the lower wealth in the system, and the effect is large if the bank being hit is an aggressive bank. However, comparing the reduction in the prices, we can see that the price of the loan type that has higher level of riskiness (larger U_k) reduces more. We can also see this from taking the difference due to the shock on both sides of (29) to get

$$\frac{p_1^h - p_1^h(v_j)}{U_1} = \frac{p_2^h - p_2^h(v_j)}{U_2} \quad \Rightarrow \quad p_1^h(v_j) - p_1^h = \frac{U1}{U_2}(p_2^h(v_j) - p_2^h).$$

So the price of the loan with a higher level of riskiness is more sensitive to a bank shock, and the effect is large if the shocks hit an aggressive bank.

Now if the shock is large enough to make bank j become insolvent, but all the other banks are not, the result is similar to the one-type case as given in (26). That is, the intetbank liabilities of the insolvent bank transmit losses to its neighbor banks, and the impact to the loan prices is large if the liabilities are large and the neighbor banks are aggressive banks.

4.2.3 After an asset shock

In this section we focus on how a shock in the default probability of one type of loans causes the change in the price of the other type of loans. Let us assume for the moment that defaults of different types of loans are independent ($\alpha = 0$). Now suppose the default probability of the type-1 loan increases from λ_1 to λ_1'. As mentioned earlier in the one-type case, $\psi_1 - (1-\lambda_1)^2$ and $1-\lambda_1-\psi_1$ are increasing in λ_1 when $\lambda_1 < 0.5$. Thus, U_1 increases as the default probability λ_1 increases. We assume further that the increase in the default risk does not cause any banks to become insolvent. Based on (33) for $k = 1$, it is clear that, the equilibrium price of type-1 loan decreases. This is due to the lower expected payoff, higher level of riskiness, and lower wealth in the system. The higher default risk of the type-1 loans also reduces the price of the type-2 loans as can be seen in (33) for $k = 2$. This cross-asset contagion comes from the wealth effect in the denominator of the type-2 loan's premium term.

Note that the cross-asset contagion always occurs as the term $\sum_{i=1}^N \theta_{i,1}/\gamma_i$ in the equilibrium price p_2^h is always positive. To see how this happens, we note that when the defaults of the two types of loans are uncorrelated ($\alpha = 0$), $\eta_{1,2} = 0$ and thus the loan demand is $\hat{\theta}_{i,k} = \bar{\theta}_{i,k}$, which is assumed to be positive for all $i = 1, \ldots, N$ and $k = 1, 2$. Now consider $\bar{\theta}_{i,k}$ as given by (13) with the subscript $k = 1, 2$. Let us assume that $e_i \gg 0$ so that the negative adjustment term in (13) is insignificant. As λ_1 increases, the type-1 loan characteristic becomes worsen as the risk ($\psi_1 - (1-\lambda_1)^2$) rises and the expected payoff ($1-\lambda_1$) declines. This makes $\bar{\theta}_{i,1}$ lower. So the demand for the type-1 loans decreases, and consequently the price of the type-1 loans has to drop to make the expected profit go up to bring the demand back to the balance. But once the price of the type-1 loans decreases, the equity value of each bank holding the type-1 loans decreases, and this reduces demand $\bar{\theta}_{i,2}$ for the type-2 loans due to the wealth effect. As a result, the price of the type-2 loans has to drop to make the expected profit higher and bring the demand back to the balance.

Now consider a little more extreme case in which the banks are divided into two non-overlapping groups, one holding only the type-1 loans and the other holding only the type-2 loans at time 0. The cross-asset contagion still occurs in this case as long as each bank has a demand for both types of the loans at time 1 before the shock. The magnitude of the effect of the cross-asset contagion from type-1 loans to type-2 loans depends, however, on the initial banks' holdings of type-1 loans at time 0, which is $\sum_{i=1}^{N} \theta_{i,1}/\gamma_i$. If initially the majority of the type-1 loans are held by aggressive banks (low γ), the impact of the cross-asset contagion from type-1 to type-2 is large, while the impact is smaller if most of the type-1 loans are held by conservative banks. The latter is unlikely if banks try to hold optimal number of loans at time 0 as aggressive banks tend to hold more loans. In Section 4.3, we discuss the cases where banks may hold only one type of loans at an equilibrium before a shock due to different expertise. In that case, the results can be different.

When the default correlation between the two types is not zero ($\alpha > 0$), the hedging demand term $\Psi_{1,2} - (1-\lambda_1)(1-\lambda_2)$ in U_2 can transmit the effect of the increase in the default probability of type-1 loans to the price of type-2 loans. However, the relationship between $\Psi_{1,2} - (1-\lambda_1)(1-\lambda_2)$ and λ_1 is not monotone for typical values of λ_1, λ_2 and α. So it is possible that the hedging demand term can strengthen or weaken the contagion effect.

4.3 Two loan types with bank expertise

In this section we assume that each bank has its own expertise in managing one particular type of loans. Let $\mathbb{N}_k$ denote the set of banks that have an expertise in managing type-k loans, $k = 1, 2$. We assume that each bank belongs to either $\mathbb{N}_1$ or $\mathbb{N}_2$, but not both. We call banks that are in $\mathbb{N}_k$ as type-k *expert banks* and those that are not in $\mathbb{N}_k$ as type-k *non-expert* banks. So the banking industry is divided into two sectors defined by $\mathbb{N}_1$ and $\mathbb{N}_2$. The cost associated with managing the loans of type k for type-k expert banks is zero, while the cost for every type-k non-expert bank is $f_k > 0$, which is the same for all non-expert banks. Let N_k denotes the number of banks in $\mathbb{N}_k$. We assume that there is at least one bank for each sector, or $N_k > 0$ for both $k = 1, 2$.

We assume further than the cost of managing loans of type k is so large that it is not optimal for type-k non-expert banks to hold type-k loans in their portfolios at an equilibrium before a shock. We also assume that the initial holdings of type-k loans for type-k non-expert banks are zero due to the high managing cost, or $\theta_{j,k} = 0$ for all $j \notin \mathbb{N}_k$. We are interested in how the loan sectors play a role on contagion risk in the banking network. Again, we consider the equilibrium prices before shocks, after a bank shock, and after an asset shock.

4.3.1 Before shocks

Because we assume that banks in sector k do not hold loans of the other types at an equilibrium before shocks, it must be that $\theta^*_{j,k} \le 0$ for $j \notin \mathbb{N}_k$. From (16), we have $\hat{\theta}_{i,k} = \bar{\theta}_{i,k}$ and $\hat{\theta}_{i,k'} = 0$ for $i \in \mathbb{N}_k$ and $k' \neq k$. Thus, the clearing condition (7) gives

$$\Theta_{S,k} = \left[\left(\sum_{i\in\mathbb{N}_k} \frac{\theta_{i,k}}{\gamma_i}\right) p_k + \sum_{i\in\mathbb{N}_k} \frac{\bar{c}_i}{\gamma_i}\right] \left(\frac{1-\lambda_k - p_k}{\psi_k^2 - (1-\lambda_k)^2}\right) - \frac{N_k}{2}\left(\frac{1-\lambda_k-\psi_k}{\psi_k^2-(1-\lambda_k)^2}\right)$$

for $k = 1, 2$. Note that we have used the fact that $\theta_{i,k} = 0$ for $i \notin \mathbb{N}_k$. As we can see, the equilibrium price for each type of loans can be determined independently as the equations for $k = 1, 2$ are decoupled. Thus, as long as banks do not have demands for loans outside their expertise, the equilibrium price of each loan type is determined based on the banks in the sector. This reduces the problem into two independent one-asset equilibrium problems. Hence, we have the equilibrium price of type-k loans is

$$p_k^h \approx 1 - \lambda_k - \frac{(\psi_k - (1-\lambda_k)^2)\Theta_{S,k} + \frac{N_k}{2}(1-\lambda_k-\psi_k)}{\sum_{i\in\mathbb{N}_k} \frac{\theta_{i,k}}{\gamma_i}(1-\lambda_k) + \sum_{i\in\mathbb{N}_k}\frac{\bar{c}_i}{\gamma_i}}.$$

As we can see, the price of type-1 loans does not depend on the default probability of type-2 loans, nor the default correlation. It does not depend on the information about the banks in sector 2 either. This holds true as long as $\theta^*_{i,2} \le 0$ for all $i \in \mathbb{N}_2$. So the contagion across banks and loan types is different from the one considered in Section 4.2.

4.3.2 After a bank shock

Suppose there is an adverse small bank shock of size v_j on bank j in sector 1. We assume that after the shock none of the banks are insolvent, and that the price is still high for the banks in sector 2 to buy type-1 loans. In this case, the new equilibrium price for type-1 loans reduces to

$$p_1^h(v_j) \approx 1 - \lambda_1 - \frac{(\psi_1 - (1-\lambda_1)^2)\Theta_{S,1} + \frac{N_1}{2}(1-\lambda_1-\psi_1)}{\sum_{i\in\mathbb{N}_1} \frac{\theta_{i,1}}{\gamma_i}(1-\lambda_1) + \sum_{i\in\mathbb{N}_1}\frac{\bar{c}_i}{\gamma_i} - \frac{v_j}{\gamma_j}}$$

while the price of type-2 loans remains the same. That is, there is no contagion from sector 1 to sector 2. The price of type-1 loans reduces because the shock reduces the equity value of bank j, causing the bank to sell off some of the loans, pushing the price down to make it more attractive for the other type-1 expert banks to increase their demands. This is the same as the one-type case.

Now let's assume that the shock v_j is large enough to make bank j insolvent, but not any other banks in the system. Assume further that at the new equilibrium prices, it is not optimal for banks in one sector to buy loans in the other sector. Under these conditions, the new equilibrium price of the type-1 loans is similar to (26) in the one-type case in which the impact on the price depends on the sizes of the interbank liabilities and the risk aversion parameters of the neighbor banks. That is, the new equilibrium is

$$p_1^h(v_j) \approx 1 - \lambda_1 - \frac{(\psi_1 - (1-\lambda_1)^2)\Theta_{S,1} + \frac{N_1}{2}(1 - \lambda_1 - \psi_1)}{\sum_{i\in\mathbb{N}_1, i\neq j} \frac{\theta_{i,1}}{\gamma_i}(1-\lambda_1) + \sum_{i\in\mathbb{N}_1, i\neq j} \frac{\bar{c}_i}{\gamma_i} - \sum_{i\in\mathbb{N}_1, i\neq j} \frac{l_{j,i}}{\gamma_i} \min\left\{ \frac{v_j - (p_1^h(v_j)\theta_{j,i} + \bar{c}_j)}{L_j}, 1 \right\}}.$$

As we can see, only the parameters describing the banking network inside sector 1 are involved.

Let us look at the price of type-2 loans. Only if there is a bank in sector 2 that is an interbank creditor of bank j, the loss of bank j can be transmitted to sector 2 through the interbank channel. This transmitted loss reduces the wealth in the sector, causing the equilibrium price for the type-2 loans to drop. The new equilibrium type-2 price is

$$p_2^h(v_j) \approx 1 - \lambda_2 - \frac{(\psi_2 - (1-\lambda_2)^2)\Theta_{S,2} + \frac{N_2}{2}(1 - \lambda_2 - \psi_2)}{\sum_{i\in\mathbb{N}_2} \frac{\theta_{i,2}}{\gamma_i}(1-\lambda_2) + \sum_{i\in\mathbb{N}_2} \frac{\bar{c}_i}{\gamma_i} - \sum_{i\in\mathbb{N}_2} \frac{l_{j,i}}{\gamma_i} \min\left\{ \frac{v_j - (p_1^h(v_j)\theta_{j,i} + \bar{c}_j)}{L_j}, 1 \right\}}.$$

In addition to the liability sizes $l_{j,i}$ and the risk aversion parameters γ_i of the interbank creditors i in sector 2, the reduction in the type-2 loan price depends also on the price of the type-1 loans after the shock or $p_1^h(v_j)$. So the more sensitive $p_1^h(v_j)$ to the shock, the higher the impact the shock has on $p_2^h(v_j)$. Because the new equilibrium price $p_1^h(v_j)$ depends on the information of all the banks in sector 1, the contagion effect from sector 1 to sector 2 depends on the information of all banks in sector 1 and how they are related. For example, if the insolvent bank j has two interbank creditors, which are bank i in sector 1 and bank i' in sector 2. Given the shock v_j, the change in the price of the type-2 loans depends not only on the information about bank j and the liability link between bank j and bank i' in sector 2, but also the liability link between bank j and bank i in sector 1 as well as the risk aversion parameter of bank i.

When this type of shock gets larger, and more type-1 expert banks become insolvent, the losses from sector 1 can be transmitted to sector 2 via the interbank liabilities between the insolvent banks in sector 1 and the banks in sector 2. So even if there is no direct interbank liability from bank j to any bank in sector 2, the loss originated from the shock on bank j may eventually affect the price of the type-2 loans if there is a liability path starting from bank j to a bank in sector 2.

Now let us consider another possible outcome from the shock v_j. Assume that after the shock, none of the banks is insolvent and the resulting equilibrium price of the type-1 loans is low enough to make it attractive to bank i' in sector 2 to hold some positive number of type-1 loans, but it is not attractive enough for the other banks in sector 2. Suppose for the moment that the defaults of type-1 and type-2 loans are uncorrelated, and thus $\eta_{1.2} = 0$. From (15) and (16) we have the optimal loan holdings of bank i' are $(\hat{\theta}_{i',1}, \hat{\theta}_{i',2}) = (\bar{\theta}_{i',1}, \bar{\theta}_{i',2})$. That is, bank i' does not change the holding in type-2 loans, but increases the holding of type-1 loans from zero to $\bar{\theta}_{i',1}$. As a consequence, the market for type-2 loans is not affected by the shock, resulting in the same equilibrium price for the type-2 loans. On the other hand, there is a loss in the equity of bank j in sector 1 and an additional demand for type-1 loans from bank i' originally from sector 2. Thus the new equilibrium price of the type-1 loans satisfies

$$\begin{aligned}\Theta_{S,1} &= \left[\left(\sum_{i\in\mathbb{N}_1}\frac{\theta_{i,1}}{\gamma_i}\right)p_1 + \sum_{i\in\mathbb{N}_1}\frac{\bar{c}_i}{\gamma_i} - \frac{v_j}{\gamma_j}\right]\left(\frac{1-\lambda_1-p_1}{\psi_1^2-(1-\lambda_1)^2}\right) - \frac{N_1}{2}\left(\frac{1-\lambda_1-\psi_1}{\psi_1^2-(1-\lambda_1)^2}\right)\\
&\quad + \left[\left(\frac{\theta_{i',2}}{\gamma_{i'}}\right)p_2 + \frac{\bar{c}_{i'}}{\gamma_{i'}}\right]\left(\frac{1-\lambda_1-f_1-p_1}{\psi_1^2-(1-\lambda_1)^2}\right) - \frac{1}{2}\left(\frac{1-\lambda_1-\psi_1}{\psi_1^2-(1-\lambda_1)^2}\right)\\
&= \left[\left(\sum_{i\in\mathbb{N}_1}\frac{\theta_{i,1}}{\gamma_i}\right)p_1 + \sum_{i\in\mathbb{N}_1}\frac{\bar{c}_i}{\gamma_i} - \frac{v_j}{\gamma_j} + \left(\frac{\theta_{i',2}}{\gamma_{i'}}\right)p_2 + \frac{\bar{c}_{i'}}{\gamma_{i'}}\right]\left(\frac{1-\lambda_1-p_1}{\psi_1^2-(1-\lambda_1)^2}\right)\\
&\quad - \frac{N_1+1}{2}\left(\frac{1-\lambda_1-\psi_1}{\psi_1^2-(1-\lambda_1)^2}\right) - \left[\left(\frac{\theta_{i',2}}{\gamma_{i'}}\right)p_2 + \frac{\bar{c}_{i'}}{\gamma_{i'}}\right]\left(\frac{f_1}{\psi_1^2-(1-\lambda_1)^2}\right)\end{aligned}$$

Using the first-order Taylor approximation as in the one-type case, we have the new equilibrium price for type-1 loans is

$$p_1^h(v_j) \approx 1-\lambda_1 - \frac{(\psi_1^2-(1-\lambda_1)^2)\Theta_{S,1} + \frac{N_1+1}{2}(1-\lambda_1-\psi_1) + \left[\left(\frac{\theta_{i',2}}{\gamma_{i'}}\right)p_2 + \frac{\bar{c}_{i'}}{\gamma_{i'}}\right]f_1}{\left(\sum_{i\in\mathbb{N}_1}\frac{\theta_{i,1}}{\gamma_i}\right)(1-\lambda_1) + \sum_{i\in\mathbb{N}_1}\frac{\bar{c}_i}{\gamma_i} - \frac{v_j}{\gamma_j} + \left[\left(\frac{\theta_{i',2}}{\gamma_{i'}}\right)p_2 + \frac{\bar{c}_{i'}}{\gamma_{i'}}\right]}. \quad (34)$$

It is easy to show that the additional demand from bank i' for the type-1 loans helps reduce the effect of the shock on the type-1 loan price given that it is optimal for the type-2 expert bank i' to enter into sector 1. Moreover, as loan defaults are uncorrelated, this does not hurt the price of the type-2 loans. To further understand this situation, let us identify the bank in sector 2 that actually is bank i'. To do this, consider a type-2 expert bank i. We can rewrite

(13) for bank i after the shock as follows:

$$\bar{\theta}_{i,1}(v_j) = \left(\frac{\theta_{i,2}p_2 + \bar{c}_i}{\gamma_i}\right)\left[\frac{(1-\lambda_1) - p_1^h(v_j) - f_1}{\psi_1 - (1-\lambda_1)^2}\right] - \frac{1}{2}\left[\frac{1-\lambda_1-\psi_1}{\psi_1 - (1-\lambda_1)^2}\right]. \tag{35}$$

As $p_1^h(v_j)$ decreases, the value of $\bar{\theta}_{i,1}(v_j)$ increases as there is no wealth effect for bank i in sector 2. So the bank that has the largest $\bar{\theta}_{i,1}(v_j)$ for $i \in \mathbb{N}_2$ is the bank i'. It is easy to see from (35) that it is the bank with the largest equity to risk aversion parameter ratio (e/γ) among the type-2 expert banks as all the banks in sector 2 has the same managing cost of f_1.

Now if loan defaults are correlated, or $\eta_{1,2} > 0$, the demand for type-1 loans from bank i' will lead to a decline in the demand for type-2 loans from bank i' due to the negative hedging demand. As the price effect is stronger than the wealth effect at the equilibrium we are interested in, the price of the type-2 loans must drop to bring the type-2 loan demand up to meet the total supply. Hence this creates a cross-asset contagion purely through the hedging demand. We do not require any interbank liabilities, nor do we require a bank to hold both types of loans at time 0 to act as a channel to transmit the effect from one type of loans to the other type of loans through the reduction in the equity value of the bank.

Now consider another alternative outcome. Suppose that the shock v_j causes a bank in sector 1 to become insolvent, and the insolvent bank has interbank liabilities with some type-2 expert banks. The resulting equilibrium prices depends on these liabilities. If banks that hold claims on the insolvent bank j are the ones with low e/γ ratios, then it is possible that the type-2 expert banks with the largest e/γ ratio will find the drop in the type-1 loan price attractive enough to buy them into the bank's balance sheet, reducing the effect on the new equilibrium price of the type-1 loans, but at the same time causing the contagion to the price of the type-2 loans due to the negative hedging demand. On the contrary, if banks that hold claims on the insolvent bank j are the ones with highest e/γ ratios, the reduction in the equity values of those type-2 expert banks could reduce the possibility for them to be the potential buyers of type-1 loans. This results in a worse outcome for the price of the type-1 loans as no new buyers from sector 2. This suggests that interbank liabilities of this type weaken the role as the potential buyers of the banks with largest e/γ.

4.3.3 After an asset shock

Suppose the default probability of the type-1 loan increases from λ_1 to λ_1'. As long as the new equilibrium price of the type-1 loans does not fall enough to attract type-2 expert banks to buy type-1 loans, and there are no losses transmitted through the liability linkages to banks in sector 2, this does not affect the equilibrium price of the type-2 loans. But once one of those scenarios occurs, the price of the type-2 loans reduces due to either the negative hedging demand, provided that $\eta_{1,2} > 0$, or the reduction in the equity values of some type-2 expert banks similar to the case of a bank shock discussed above.

The above discussion leads to an interesting policy implication. Suppose that the costs of managing loans are low, but the regulator would like to separate banks and loan markets into non-overlapping sectors to limit the effect of contagion. As a consequence, the regulator may allow banks to choose their areas of expertise or sectors where they can run their businesses as usual. However, banks need to pay a huge regulatory fee to do the business outside their selected areas of expertise. When a sector is hit by a small shock, the banks in the sector can function as potential buyers to self-rescue the sector from fire sales. When the shock is large, and there are not many banks in the sector that can function as the potential buyers, the regulator can initiate the self-rescue mission by allowing the secondary potential buyers from the other healthy sectors to step in and buy the assets, reducing the effect of fire sales in the failing sector. The regulator can choose the right value of $f_{i,k}$ to allow enough funds from other sectors to flow into the failing sector, providing support to the loan price in the failing sector. At the same time the regulator needs to avoid the unintended contagion effects due to the negative hedging demands.

Once the cross-sector rescue mission has been accomplished, the healthy sector is now contaminated by the fire sale loans, and cannot function as secondary potential buyers for the next crisis. So the regulator should use this as a temporary solution to reduce the effect of fire sales, and start to bring everything back to normal and be ready for the next crisis.

5 Conclusion

When an adverse shock hits a bank, causing it to become insolvent, the bank needs to sell all of its assets, the majority of which are illiquid loans. This can cause the loan prices to drop, reducing the mark-to-market values of other banks holding the same types of loans. The loss of the insolvent bank can also be transmitted to other banks through the interbank liability linkages, reducing the net worth of its neighbor banks. The banks affected by these two channels of contagion will re-adjust their portfolios in response to lower equity values, and start to sell more illiquid loans into the markets. If banks are highly connected either through the liability linkages or the common loan holdings, then most of the banks in the system will suffer from the losses and cannot function as the potential buyers, reducing the self-rescue ability of the system.

We study a financial system in which banks in the system may create self-rescue ability. We find that aggressive banks can become good potential buyers if they are not affected by a shock as they are willing to buy a large amount of loans given a small discount. At the same time, they can become fire sale initiators even if they are hit by a relatively small shock. Interbank liabilities between these aggressive banks can also amplify the contagion effect and the effect of fire sales as they adjust the portfolios markedly following losses in their equity values. So it is better to avoid having interbank liabilities between those aggressive banks. We also find that prices of loans that have higher risk are more sensitive to a shock in the system. So having aggressive banks holding these high risk loans would accelerate the contagion effect, once it occurs. Unfortunately, aggressive banks tend to hold a large amount of loans, including the high risk loans, so we need some regulatory policies to help reduce the potential damages caused by these aggressive banks.

Contagion across loan types can occur from many channels. A shock to one particular bank may trigger a fire sale of one loan type, lowering the loan price. Banks that hold the same type of loans will lose their equity values and start to sell loans of other types in their portfolios to re-adjust their portfolios' risk-adjusted returns. This creates the cross-asset contagion. Alternatively, a drop of the price of one loan type makes it more attractive to healthy banks to buy the loans. As loan defaults are positively correlated, the substitute effect creates a negative hedging demand, requiring the banks to reduce the holdings of certain types of loans in their portfolios upon buying another type of loans.

Finally, we study the system in which banks and loan markets are separated into sectors based on their areas of expertise defined by the cost of managing loans. We find that small shocks in one sector do not cause contagion to the others as long as the interbank liabilities between the sectors are not available and the cost for entering an area outside of the banks' expertise is sufficiently high. In this case, the banks in the sector experiencing a small shock need to act as potential buyers for their own sectors. Once the shock is large, causing the price to drop enough, then banks from other sectors may function as potential buyers for the failing sector. Based on this observation, we propose a policy that separates banks and loans into sectors, limiting the contagion effects between groups of banks and types of loans. At the same time, we create secondary potential buyers that are ready to step in and save the failing sector when it is most needed. This type of policy can be achieved by imposing regulatory fees that keep them separated during good times, and allow them to rescue their peers during bad times, creating a self-rescue system.

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