Optimal liquidity control and systemic risk in an interbank network with liquidity shocks and regime-dependent interconnectedness *

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Abstract

We propose a novel interbank network model in which banks face systemic liquidity shocks, flight-to-quality liquidity flows, and collapses of the interbank network during crises, and study their impacts on the optimal liquidity control and the systemic risk of the interbank network. We find that banks respond to negative shocks by holding positive *precautionary liquidity*, but once the shock size is sufficiently large, the benefit of precautionary liquidity reduces, and banks lower their precautionary liquidity. Lending (borrowing) banks also hold positive (negative) *interbank liquidity provision*. Banks hold more provision for more interconnected networks, but when the network is too interconnected, it is too costly to hold large provision, causing banks to lower the provision. On the contrary, a higher degree of the flight-to-quality effect tends to make banks act more aggressively on both precautionary liquidity and interbank provision. As a result, the systemic risk tends to increase in the size of the negative shock, but is quite insensitive to the degree of the flight-to-quality effect. Our analysis shows that the systemic risk increases if the interbank market collapses or becomes too interconnected during crises. Rewards and penalties from regulators can help reduce the systemic risk, but they come with a cost and have different implications on the banks' optimal policies. (JEL C44, C73, D81, D85, E02, G01, G21, G28, L14)

Keywords: Liquidity shock, interbank interconnectedness, flight-to-quality, systemic risk, precautionary liquidity, interbank liquidity provision, regime switching, stochastic control.

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1 Introduction

Interbank borrowing and lending markets are one of the most important parts of the financial systems. Banks can lend and borrow money with one another in the markets in order to manage their short-term liquidity and meet some regulatory requirements, for instance, the reserve requirement. Loan maturities in the markets are mostly shorter than one week, the main transactions of which are being overnight. Therefore, if some banks suffer the liquidity shocks, the impact will spillover through the markets almost instantaneously, and it could subsequently damage the financial systems as well as the real economy. The global financial crisis during 2007-2008 is an obvious evidence that many financial institutions were in liquidity shortage and volumes traded in the markets were sharply reduced. This remarkably triggered the systemic liquidity risk, causing further damages to the financial markets.

Liquidity shocks during a financial crisis usually greatly reduce the level of interconnectedness in the interbank market due to liquidity hoarding and increased counterparty risk, and leave banks to manage their liquidity outside the interbank market. It is questionable whether the reduction in the interconnectedness would help moderate the liquidity risk and stabilize the financial system at the time when the liquidity is most needed. In addition, a banking panic can cause withdrawals of funds which will worsen the situation. There are two main questions we try to answer in this research. The first question is what is the optimal policy for liquidity control of a bank in the banking system that is subject to different sources of liquidity risk including (i) the idiosyncratic shocks from daily deposits and withdrawals, (ii) systemic shocks such as bank runs, (iii) reduction in the interconnectedness in the interbank market during a crisis, and (iv) panic-triggered liquidity outflows following large shocks? An answer to this question will provide some useful guideline for banks to manage their liquidity risk, not only in the crisis times, but also during the normal times.

The second question concerns the risk of the banking system: how much each of the four sources of liquidity risk contributes to the increase in the systemic risk of the banking system? Regulators and policy makers should plan for possibilities of such liquidity risk, and develop related preventive measures and risk mitigation plans. The answer of this research question would provide some useful insights for the regulators and policy makers in developing effective policies.

To answer the main research questions, we develop a novel interbank liquidity network model that accounts for all of the four sources of liquidity risk. This extends greatly from the optimal liquidity control literature in which banks are only subject to liquidity risk from daily deposits and withdrawals. In particular, we allow the market conditions to switch between a normal regime and a crisis regime in which the level of interconnectedness differs. This enables us to model a reduction in the interconnectedness in the interbank market during a crisis regime, and study its effect on the optimal liquidity control policy of banks as well as the effect on systemic risk of the banking system. Our interconnectedness is measured by two factors: the interbank exposures (transaction sizes) between banks, and the network topology (connectivity). We also allow random liquidity shocks which can cause a sharp reduction in the banks' liquidity. We consider two types of shocks: within-regime shocks and regime-transition shocks. The within-regime shocks are shocks that do not result in a change in the market regime, while regime-transition shocks not only cause a reduction in the banks' liquidity, but also lead to a change in the market regime. The latter can be used to model a systemic bank run which is followed by a change from a normal market to a crisis. We study both types of shocks and their impacts on the banks' optimal controls and systemic risk. Panic-triggered liquidity flows are also included in our model to represent continuous outflows from withdrawals during a crisis period.

Our model can also be used as a tool for conducting stress testing and scenario analysis. Most central banks have access to data such as interbank transactions and banks' monetary reserves and short-term liquidity. The central banks can use such data to fit our model and test various scenarios on the fitted interbank network to measure the stability and fragility of the network under different assumptions such as the size of the shock, the network topology during a crisis, or the severity level of the panic. This will, for example, help the central banks estimate how much the system can absorb shocks and identify banks that are systemically important. It can also suggest which network topology and level of interbank exposure are appropriate during a normal and a crisis regime.

The rest of the paper is organized as follows: Section 2 provides a literature review. Section 3 describes our interbank network model, formulates the optimal control problems, and characterizes the optimal control policy. We study a special case of a large homogeneous complete network with one regime in Section 4. We extend our analysis to a core-periphery network with numerical examples in Section 5. Section 6 analyzes the implications on the systemic risk of the interbank network. Section 7 discusses policy implications and concludes the study.

2 Literature review

Interbank markets play a key role in liquidity management for the banking system, but at the same time create channels of shock transmission between banks. A liquidity shock at one bank can spread to the other banks in the network, which can eventually lead to a systemic loss. There is a growing literature about the relation between the stability of financial systems and the degree of network interconnections. Allen and Gale (2000) and Freixas et al. (2000) propose theoretical frameworks to explain contagion in financial networks in which banks are interconnected via interbank exposures. Their findings suggest that the fragility of the financial system depends on network structure. In particular, Allen and Gale (2000) conclude that the complete (or fully-linked) network helps alleviate financial vulnerability as the risk sharing on more counterparties reduces the magnitude of the default contagion and non-distressed banks can provide excess liquidity to the system. Nier et al. (2007) also show that a high degree of interconnections in the banking system could enhance the ability to absorb shock. Other studies, however, suggest that densely interbank interconnectedness could also promote instability in the financial system due to adverse amplification of shock. So limited interconnectedness could handle the distress propagation in banking system (see, for example, Gai et al. (2011), Mistrulli (2011), and Blume et al. (2011, 2013)). Acemoglu et al. (2015) analyze the network stability conditional on a shock, and show that when the shock is small, a more densely connected network provides better stability due to risk-sharing benefits, while when the shock is large, the dense connection creates fragility of the network by propagating shocks throughout the network. This is consistent with the experiment of Gai and Kapadia (2010) who report that financial networks are robust-but-fragile. Stiglitz (2010) has a similar conclusion that financial integration can increase the risk of adverse contagion in the event of a large negative shock, and views the diversification and contagion as two different sides of the same coin. Hence, the network topology seems to play a crucial role on the system stability, and its role can be changed depending on the size of the shock. For a recent literature survey related to financial contagion, see Glasserman and Young (2016).

Our study also focuses on shifts in the network's interconnectedness between normal and crisis regimes. Martinez-Jaramillo et al. (2014) find that the topology of the exposures network in the Maxican interbank market changed after the failure of Lehman Brothers. Affinito and Pozzolo (2017) use the eigenvector centrality as a measure of interconnectedness, which depends on the number of the strength of the connections, and conclude that after the 2007 liquidity crisis, there was a marked reduction of the relative interconnectedness of the Italian banking sector. In addition, they find that the system recovered its patterns in the following years. Brunetti et al. (2019) study the changes in the interconnectedness between European banks around the 2008 financial crisis. They find that the interconnectedness of the physical network based on the interbank transactions markedly decreased during the crisis. The results are robust to the measures of the interconnectedness, which include the degree, closeness, clustering coefficient, eigenvector centrality, and largest strongly connected component. Similarly, Calomiris et al. (2019) report that the U.S. interbank network was collapsed during the Great Depression. They also find that the establishment of the Federal Reserve reduced banks' incentive to maintain capital and cash buffer as they expected helps from the Fed during the critical time, and this likely contributed to the banking system's vulnerability to contagion during the Depression. Gofman (2017) studies the impact of interconnectedness restriction on the system stability and efficiency, and finds that restricting the interconnectedness reduces trading efficiency but increases the stability, although the results are not monotonic. However, Chiu et al. (2020) develop a model of interbank relationship in which banks could reduce the cost of trading liquidity and insure against liquidity shocks by maintaining their relationship. They find that interbank relationship can explain some anomalies. This may suggest that banks are possible to maintain certain relationship even in the bad times. In our model, we account for changes in the network's interconnectedness between the normal regime and the crisis regime. We model the changes by two components: the overall exposures and the network topology. It is interesting to see how the changes in the interconnectedness in the interbank markets contribute to the systemic risk when a liquidity crisis occurs.

Another focus of our study is on the analyses of optimal liquidity control in banking system and the relation between the control policy and the interbank network. Fouque and Ichiba (2013) introduce a dynamic stochastic model for interbank lending in the continuous time framework. They link interbank lending activities into a dynamic of monetary reserves that follows a Feller diffusion process. In their model, banks are assumed to control their monetary reserves by lending and borrowing money in the interbank market. The decision to borrow or to lend money depends on the relative level of banks' monetary reserves. With identical lending preferences, the model reduces to a mean-field dynamic in which the lending decision depends on the level of banks' monetary reserves relative to the average of the monetary reserve in the system.

There are many extensions to the the mean-field models for interbank lending. Bo and Capponi (2015) and Borovykh et al. (2018) work on derivation of weak limits of measure-valued processes to approximate systemic risk indicators when the number of banks in the system is large and banks are subject to jump risk, but unlike Fouque and Ichiba (2013), Bo and Capponi (2015) and Borovykh et al. (2018) do not allow banks to optimally control their monetary reserves in their settings. Carmona et al. (2015, 2018), and Sun (2017, 2018) study optimal monetary reserve control problems in a banking system, in which each bank tries to control its monetary reserve to minimize the distance between its reserve and the average of monetary reserve in the system. Nevertheless, under the mean-field assumption, reserves tend to convert to the average of the reserve in the system. This is not a desirable property as different banks may not necessarily adjust their reserve levels close to the level of the reserve average in the system. Bordeleau and Graham (2010) show that there are optimal levels of banks' liquidity for profit maximization which depend on the banks' business models. Minimum reserve requirements are also different between banks. When the average reserve level of the system drops close to or below the requirement level, it is not sensible that banks should try to control reserves to meet the average level. In our study, we relax this assumption and allow banks to control their liquidity towards their own targets.

Due to the drawbacks of mean-field models, Capponi et al. (2020) derive a set of weak limiting measured-value processes to approximate systemic risk indicators under a dynamic model of banks' monetary reserves extended from the mean-field case. In their model, heterogeneous bank clusters are allowed to capture different characteristics (e.g. rates of reserve adjustment) and network topologies are incorporated. The lending decision in this model depends on whether the banks' monetary reserves stay above or below the required reserves, instead of the average of monetary reserves in the market. In other words, if the banks' monetary reserves are higher (lower) than their required reserves, the banks are prone to lend (borrow) the reserves in the interbank market. However, the model of Capponi et al. (2020) assumes that banks simply follow this rule and there is no room for banks to adjust their reserves, for instance, in response to worsen financial conditions. In our study, we allow banks to adjust their liquidity levels in addition to the relationship in the interbank markets.

One important factor that is missing from the literature is the presence of systemic liquidity shocks to the interbank markets. These shocks can be used to model, for example, a large deposit run-off during a financial crisis. Lim and Watewai (2012) develop a regime switching model for stock markets in which shocks on asset prices can occur simultaneously with regime shifts. We follow this framework, and allow the regime to change following an arrival of a systemic shock in our banking system. There are many works that use Markov regime switching and jump processes to model changes in the market regimes and event risk, respectively, in stock markets. See, for example, Ang and Bekaert (2002) for a model with regime shifts, Das and Uppal (2004) for a model with Poisson jumps, and Aït-Sahalia et al. (2015) for a model with Hawkes processes.

3 Optimal liquidity control in an interbank network

In this section we first describe the interbank network model of banks' liquidity that is subject to liquidity shocks and regime-dependent interconnectedness. Then we formulate the optimal liquidity control problem and define the equilibrium conditions of the liquidity network. We characterize the optimality condition using the stochastic control theory.

3.1 Interbank liquidity network model

Consider an interbank liquidity network consisting of M banks. Each bank i has a liquidity of $X_i(t)$ at time t. Changes in the liquidity of bank i come from 5 different sources: (i) bank's own control, (ii) interbank borrowing and lending, (iii) idiosyncrasy in the bank's liquidity (such as changes due to daily deposits and withdrawals), (iv) liquidility shocks, and (v) panic-triggered flows. In particular, the liquidity of bank i is given by

$$X_i(t) = X_i(0) + U_i(t) + I_i(t) + B_i(t) + S_i(t) + P_i(t) \tag{1}$$

where U, I, B, S and P represent the accumulated liquidity changes from the five sources, respectively. The banking system is also subject to regime shifts. Let $Y(t) \in \{1, \ldots, K\}$ denote the regime of the banking system at time t. These regimes could represent the state of the economy for the banking system such as a normal state and a crisis state. In this study, we assume that the regime process $Y(t)$ follows a $K-$state continuous-time Markov chain with transition rate $\lambda(y,z)$ from regime y to regime z.

At each time t, bank i can control its liquidity by raising or removing the liquidity at a rate $\alpha_i(t) \in \mathbb{R}$:

$$dU_i(t) = \alpha_i(t)dt. \tag{2}$$

For example, the bank may raise its short-term liquidity by borrowing from the central bank ($\alpha > 0$).

The interbank borrowing and lending is one of the most important channels for short-term liquidity management. We assume that each bank i has a regime-dependent liquidity target $\theta_i(y) > 0$ which represents an optimal level of liquidity in terms of the bank's profitability for bank i when the regime is y. This is consistent with the findings from Bordeleau and Graham (2010) who show empirically that there is an optimal level of liquidity for which the profit is maximized, and that the optimal level of liquidity varies depending on the bank's business model and the state of the economy. We assume also that banks lend and borrow money among each other based on three factors: their liquidity levels relative to their targets, their lending preference, and the overall network exposure. More specifically, let $\pi_{i,j}(y) \geq 0$ denote the lending preference between bank i and bank j in regime y where $\pi_{i,j} = \pi_{j,i}$ and $\pi_{i,i} = 0$, and $\zeta(y) \geq 0$ denote the level of the overall network exposure in regime y. The lending preference represents the relationships between banks in the interbank market, which constitutes a network topology. A large value of π indicates a good relationship between the banks (see, for example, Bech and Atalay (2010), and Chiu et al. (2020) for evidence of banks maintaining relationships in the interbank markets.) The exposure $\zeta(y)$ represents the overall level of the interbank activities in regime y. Both of the lending preference π and overall exposure ζ jointly determine the level of the interconnectedness of the network. The regime-dependent interconnectedness allows us to model, for example, a high level of interbank activities with dense linkages in a good economy, and a lower level of activities with sparse linkages during crises due to liquidity hoarding and increased counterparty risk (see Brunetti et al. (2019) for an evidence of the reduction in the interbank activities during the 2008 financial crisis.) In summary, the rate at which bank j lends to bank i at time t when the regime is y is given by

$$\zeta(y)\pi_{i,j}(y)\Big\{[X_j(t) - \theta_j(y)] - [X_i(t) - \theta_i(y)]\Big\}. \tag{3}$$

Observe that bank j will lend to (borrow from) bank i if bank j has more (less) excess liquidity $(X_j - \theta_j)$ than bank i.[1] Hence, the interbank borrowing and lending affects the liquidity of bank

[1]One might argue that there should be no interbank lending between bank i and bank j if both have positive or negative excess liquidity. In our study, we assume that banks give a high priority to maintaining a good relationship with their peers (if $\pi_{i,j} > 0$) and thus banks with a better condition will help banks with a worse condition regardless

i as follows:

$$dI_i(t) = \zeta(Y(t)) \sum_{j=1}^{M} \pi_{i,j}(Y(t)) \Big\{ [X_j(t) - \theta_j(Y(t))] - [X_i(t) - \theta_i(Y(t))] \Big\} dt. \tag{4}$$

Given $\pi_{i,j} = \pi_{j,i}$, the interbank market clearing condition $\sum_{i=1}^{M} dI_i(t) = 0$ is always met. We normalize the values of π so that $\max_{i,j}\{\pi_{i,j}\} = 1$ to avoid the identification problem between π and ζ.

The third source of the liquidity changes is the idiosyncrasy of the bank, reflecting, for example, daily deposits and withdrawals of retail customers. The idiosyncratic term of bank i is given by

$$dB_i(t) = \sigma_i(Y(t))\sqrt{X_i(t)}dW_i(t) \tag{5}$$

where $\sigma_i(y) > 0$ is a regime-dependent volatility parameter, and $W_i(t), i = 1, \ldots, M$ are independent standard Brownian motions.

The liquidity of each bank is subject to systemic shocks, which can be due to, for example, bank runs. There are two types of shocks: within-regime shocks and regime-transition shocks. The within-regime shocks are shocks to liquidity within each regime, while regime-transition shocks are shocks to liquidity that also trigger a regime transition. Let $N_z(t)$ denote a Markov-modulated Poisson process, the arrival intensity of which is $\lambda(y, z)$ when the regime is $Y(t^-) = y$. An arrival from $N_z(t)$ when $Y(t^-) = y$ causes a jump in the liquidity of bank i of size $\eta_i(y, z)X(t^-)$ and causes a transition from the current regime y to regime z if $z \neq y$ (regime-transition shocks), but does not cause any regime shift when $z = y$ (within-regime shocks). Note that $\lambda(y, z)$ is also the transition rate of the regime process $Y(t)$ when $y \neq z$. We assume that these Poisson processes are independent given the regime $Y(t^-)$. The changes in liquidity due to systemic shocks for bank i are given by

$$dS_i(t) = \sum_{z=1}^{K} \eta_i(Y(t^-), z)X_i(t^-)dN_z(t). \tag{6}$$

This set of Poisson processes enables us to model an event such as a systemic bank run with negative shocks to banks' liquidity, which is followed immediately by a transition to a crisis regime. We assume that the relative jump size $\eta_i(y, z)$ is a constant in $(-1, \infty)$.

The last source of liquidity changes is from the panic-triggered flows. As the name suggested, these flows of liquidity are caused by panic. For example, large withdrawals in a systemic bank run can cause a panic to other depositors who consequently start to withdraw their money. At the same time, borrowing firms may start to use their remaining credit lines. However, banks

of the sign of their excess liquidity. After all banks can use their own control α to adjust their preferred direction of liquidity changes.

with high liquidity may receive more deposits due to flight-to-quality effects. We model these liquidity flows as follows

$$dP_i(t) = \mu_i(Y(t))[X_i(t) - \theta_i(Y(t))]dt \tag{7}$$

where $\mu_i(y) \geq 0$ is the proportional rate of the flow to bank i when the regime is y. Note that a positive value of μ_i during the crisis regime can be use to imply a flight-to-quality effect which causes a reduction in the liquidity of the banks whose excess liquidity is negative, and an increase in the liquidity of the banks whose excess liquidity is positive.

Putting everything together, we have the following dynamic of the liquidity of bank i

$$\begin{aligned} dX_i(t) = {} & \alpha_i(t)dt + \zeta(Y(t)) \sum_{j=1}^{M} \pi_{i,j}(Y(t)) \Big\{ [X_j(t) - \theta_j(Y(t))] - [X_i(t) - \theta_i(Y(t))] \Big\} dt \\ & + \sigma_i(Y(t)) \sqrt{X_i(t)} dW_i(t) + \sum_{z=1}^{K} \eta_i(Y(t^-), z) X_i(t^-) dN_z(t) \\ & + \mu_i(Y(t))[X_i(t) - \theta_i(Y(t))]dt. \end{aligned} \tag{8}$$

Finally, we assume that the Brownian motions $W_i(t), i = 1, \dots, M$ are independent of the Poisson processes $N_z(t), z = 1, \dots, K$.

The effect of liquidity hoarding during a crisis regime modelled by the regime-dependent interconnectedness together with the flight-to-quality effect can jeopardize banks with low liquidity during a crisis regime. One of our focuses is to study the impacts of these effects on the banks' optimal liquidity controls in both normal and crisis regimes, as well as the corresponding systemic risk of the banking system.

3.2 Liquidity control problems

We assume that each bank i controls its liquidity through α_i to minimize its expected discounted running cost over an infinite horizon. To proceed, define a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ that satisfies usual conditions on $t \in [0, \infty)$ on which $M-$dimensional standard Brownian motion $W(t) = [W_1(t), \dots, W_M(t)]'$ and $K-$dimensional Markov modulated Poisson processes $N(t) = [N_1(t), \dots, N_K(t)]'$ are defined. Let $X(t) = [X_1(t), \dots, X_M(t)]'$ denote the vector of banks' liquidity. Let $\alpha_{-i} = [\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_M]'$ denote the vector of strategies of all bank $j \neq i$. The problem of each bank i is to find the best response strategy α_i that minimizes the expected total discounted cost:

$$\begin{aligned} & J_i(\alpha_i; \alpha_{-i}, x, y) \\ & = \mathbb{E}\left[\int_0^\infty e^{-\delta_i(Y(t))t} f_i(\alpha_i(t), X_i(t), Y(t)) dt \,\middle|\, \alpha_{-i}, X(0) = x, Y(0) = y \right] \end{aligned} \tag{9}$$

where $X(t)$ follows the dynamic given by (8) under the strategy (α_i, α_{-i}), $\delta_i(y) > 0$ is the discount rate in regime y for bank i, and f_i is the running cost for bank i and is given by

$$f_i(\alpha_i, x_i, y) = \frac{1}{2}\alpha_i^2 - b_i(y)\alpha_i(\theta_i(y) - x_i) + \frac{q_i(y)}{2}(\theta_i(y) - x_i)^2 \tag{10}$$

where $b_i(y) \geq 0$ with $b_i(y)^2 < q_i(y)$ to ensure that f_i is strictly convex in (α_i, x_i) for each regime $y \in \{1, \ldots, K\}$. The first term in (10) represents the cost of adding or removing liquidity through borrowing from or lending to a party outside the interbank market (e.g. borrowing from the central bank). The second term represents the incentive of the bank to control the liquidity closer to the target θ_i. For example, when $X_i(t) < \theta_i(Y(t))$, raising the liquidity with $\alpha_i(t) > 0$ can reduce the running cost. To give such an incentive, the regulator or the central bank may impose lower fees on transactions that could raise the bank's liquidity such as lowering a lending rate with the central bank. The regulator or the central bank may prefer that banks maintain their liquidity at their ideal liquidity levels θ to generate the highest (risk-adjusted) profits to reduce the insolvency risk. The third term represents the loss of profit due to a deviation from the best target θ. On one hand, holding too less liquidity increases the credit risk, and hence it could lead to a higher expected cost of bankruptcy, as well as a higher financing cost. On the other hand, holding too much liquidity creates opportunity costs as banks have less ability to extend credit or invest in higher-yield assets. A quadratic running cost is also used by Sun (2017, 2018) and Carmona et al. (2015, 2018) in the context of mean-field models. Note that under the assumption that $b_i(y)^2 < q_i(y)$, $f_i(\alpha_i, x_i, y) \geq 0$, and $f_i(\alpha_i, x_i, y) = 0$ if, and only if, $x_i = \theta_i(y)$ and $\alpha_i = 0$. So banks have to find a balance between keeping their liquidity levels close to their targets and keeping the cost of controlling their liquidity as low as possible.

Given the strategies of the other banks α_{-i}, the optimal strategy for bank i denoted by $\hat{\alpha}_i$ can be derived from the Hamilton-Jacobi-Bellman (HJB) equation using the stochastic control theory, which will be given below. To further characterize an equilibrium behavior of the optimal control $\hat{\alpha} = [\hat{\alpha}_1, \ldots, \hat{\alpha}_M]'$ and their associated liquidity processes $\hat{X}(t) = [\hat{X}_1(t), \ldots, \hat{X}_M(t)]'$, we focus on the Markov-Nash equilibrium in which only Markov strategies are considered (Sun (2017, 2018) and Carmona et al. (2015) also considers a Markov-Nash equilibrium). In particular, let $\mathcal{A}_i$ denote the set of all admissible Markov policies of α_i satisfying the integrability condition

$$\mathbb{E}\left[\int_0^T |\alpha_i(t)|^2 dt\right] < \infty, \qquad \forall T > 0. \tag{11}$$

The policy $\hat{\alpha} = [\hat{\alpha}_1, \ldots, \hat{\alpha}_M]'$ with $\hat{\alpha}_j \in \mathcal{A}_j, j = 1, \ldots, M$ is a Markov-Nash equilibrium if

$$J_i(\hat{\alpha}_i; \hat{\alpha}_{-i}, x, y) = \inf_{\alpha_i \in \mathcal{A}_i} J_i(\alpha_i; \hat{\alpha}_{-i}, x, y) \tag{12}$$

for all $i = 1, \ldots, M$ and $(x, y) \in [0, \infty)^M \times \{1, 2, \ldots, K\}$.

3.3 Optimality conditions

First, we write the dynamic of vectors of liquidity levels $X(t) = [X_1(t), \ldots, X_M(t)]'$ as follows:

$$
\begin{aligned}
dX(t) &= \alpha(t)dt + \zeta(Y(t))\Pi(Y(t))\left[X(t) - \theta(Y(t))\right]dt + \sigma(Y(t))D(X(t))dW(t) \\
&\quad + \sum_{z=1}^{K} \eta(Y(t^-), z)X(t^-)dN_z(t) + \mu(Y(t))[X(t) - \theta(Y(t))]dt
\end{aligned} \tag{13}
$$

where

$$
\begin{aligned}
\alpha(t) &= [\alpha_1(t), \ldots, \alpha_M(t)]' \\
\Pi(y) &= \begin{bmatrix} -\sum_{j\neq 1}\pi_{1,j}(y) & \pi_{1,2}(y) & \cdots & \pi_{1,M}(y) \\ \pi_{2,1}(y) & -\sum_{j\neq 2}\pi_{2,j}(y) & \cdots & \pi_{2,M}(y) \\ \vdots & & \ddots & \\ \pi_{M,1}(y) & \pi_{M,2}(y) & \cdots & -\sum_{j\neq M}\pi_{M,j}(y) \end{bmatrix} \\
\theta(y) &= [\theta_1(y), \ldots, \theta_M(y)]' \\
\sigma(y) &= \begin{bmatrix} \sigma_1(y) & 0 & \cdots & 0 \\ 0 & \sigma_2(y) & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \sigma_M(y) \end{bmatrix} \\
D(x) &= \begin{bmatrix} \sqrt{x_1} & 0 & \cdots & 0 \\ 0 & \sqrt{x_2} & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \sqrt{x_M} \end{bmatrix} \\
\eta(y,z) &= \begin{bmatrix} \eta_1(y,z) & 0 & \cdots & 0 \\ 0 & \eta_2(y,z) & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \eta_M(y,z) \end{bmatrix} \\
\mu(y) &= \begin{bmatrix} \mu_1(y) & 0 & \cdots & 0 \\ 0 & \mu_2(y) & \cdots & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \mu_M(y) \end{bmatrix}.
\end{aligned}
$$

Define the value function of bank i at a Markov-Nash equilibrium by

$$V_i(x,y) = J_i(\hat{\alpha}_i; \hat{\alpha}_{-i}, x, y) \tag{14}$$

where $\hat{\alpha}_1, \ldots, \hat{\alpha}_M$ are the optimal controls at the equilibrium. The optimality conditions for the value functions $V_i(x,y)$ for each $i = 1, \ldots, M$ and $(x,y) \in [0,\infty)^M \times \{1, \ldots, K\}$ are given by the Hamilton-Jacobi-Bellman (HJB) equation:

$$\begin{aligned}\min_{\alpha_i} \Bigg\{ & \nabla_x V_i(x,y)' \left[\hat{\alpha}_{(i,-i)}(x,y) + \zeta(y)\Pi(y)(x - \theta(y)) + \mu(y)(x - \theta(y))\right] \\ & + \frac{1}{2} tr \left(\nabla^2_{xx} V_i(x,y)\sigma(y)^2 D(x)^2\right) \\ & + \sum_{z=1}^{M} \lambda(y,z) \left[V_i(x + \eta(y,z)x, z) - V_i(x,y)\right] \\ & + \frac{1}{2}\alpha_i^2 - b_i(y)\alpha_i[\theta_i(y) - x_i] + \frac{1}{2} q_i(y)[x_i - \theta_i(y)]^2 - \delta_i(y) V_i(x,y) \Bigg\} = 0 \end{aligned} \tag{15}$$

where the values of $V_i(x,y)$ have to be solved simultaneously for all $i \in \{1, \ldots, M\}$ and $(x,y) \in [0,\infty)^M \times \{1, \ldots, K\}$, and $\hat{\alpha}_{(i,-i)}(x,y) = [\hat{\alpha}_1(x,y), \ldots, \hat{\alpha}_{i-1}(x,y), \alpha_i, \hat{\alpha}_{i+1}(x,y), \ldots, \hat{\alpha}_M(x,y)]'$. The optimal liquidity control for bank i when $X(t) = x$ and $Y(t) = y$ is given by

$$\hat{\alpha}_i(x,y) = b_i(y)(\theta_i(y) - x_i) - \frac{\partial V_i(x,y)}{\partial x_i}. \tag{16}$$

The first term is to minimize the current running cost, while the second term is to minimize the future cost. Observe that, in general, the optimal control of bank i depends on the liquidity level of every bank in the system, as well as the current regime.

It can be shown that the value function of each bank i is of the following form:

$$V_i(x,y) = \frac{1}{2} A_i(y) + B_i(y)' x + \frac{1}{2} x' C_i(y) x \tag{17}$$

where $A_i(y) \in \mathbb{R}, B_i(y) \in \mathbb{R}^M$ and $C_i(y)$ is an $M-$dimensional symmetric matrix for $y = 1, \ldots, K$. Substituting $V_i(x,y)$ in (17) into the HJB equation (15), we have $A_i(y), B_i(y)$ and $C_i(y)$ are the solution of the following system of equations for $i = 1, \dots, M$ and $y = 1, \dots, K$:

$$
\begin{aligned}
0 =& \frac{1}{2}\sum_{z\neq y}\lambda(y,z)\left[A_i(z)-A_i(y)\right] - \frac{1}{2}\delta_i(y)A_i(y) + \theta(y)'\left[b(y)-\zeta(y)\Pi(y)-\mu(y)\right]B_i(y) \\
& - B_i(y)'\left[\sum_{j\neq i} e_{jj}B_j(y)\right] - \frac{1}{2}B_i(y)'e_{ii}B_i(y) - \frac{1}{2}\theta_i(y)^2\left[b_i(y)^2 - q_i(y)\right] \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
0 =& -\left[\sum_{j\neq i} C_j(y)e_{jj}\right]B_j(y) - C_i(y)\left[\sum_{j\neq i} e_{jj}B_j(y)\right] - C_i(y)e_{ii}B_i(y) \\
& + \sum_{z=1}^{M}\lambda(y,z)\left[(I+\eta(y,z))B_i(z) - B_i(y)\right] - \left[b(y)-\zeta(y)\Pi(y)-\mu(y)\right]'B_i(y) - \delta_i(y)B_i(y) \\
& + \frac{1}{2}\sigma(y)^2\sum_{j=1}^{M} e_{jj}C_i(y)e_{jj}\mathbf{1} + C_i(y)\left[b(y)-\zeta(y)\Pi(y)-\mu(y)\right]\theta(y) + \left[b(y)^2 - q(y)\right]e_{ii}\theta(y)
\end{aligned}
$$

(19)

$$
\begin{aligned}
0 =& \frac{1}{2}C_i(y)e_{ii}C_i(y) - C_i(y)\sum_{j=1}^{M} e_{jj}C_j(y) + \frac{1}{2}\sum_{z=1}^{K}\lambda(y,z)\left[(I+\eta(y,z))'C_i(z)(I+\eta(y,z)) - C_i(y)\right] \\
& - C_i(y)\left[b(y)-\zeta(y)\Pi(y)-\mu(y)+\frac{1}{2}\delta_i(y)I\right] - \frac{1}{2}\left[b(y)^2-q(y)\right]e_{ii} \qquad (20)
\end{aligned}
$$

where e_{jj} is an $M\times M$ matrix of zeros except the (j,j) entry being one, $\mathbf{1}$ is an $M-$dimensional vector of ones, $b(y)$ and $q(y)$ are the diagonal matrices with diagonal elements being $b_1(y), \dots, b_M(y)$, and $q_1(y), \dots, q_M(y)$, respectively.

The system of equations (18) - (20) for $i \in \{1, \dots, M\}, y \in \{1, \dots, K\}$ cannot be solved explicitly in the general case. To study the optimal control policy and the equilibrium of the network, we next consider a special case in which the optimal control at the equilibrium can be solved explicitly. Then we use a numerical method to solve (18) - (20) for a more complex case.

4 Large homogeneous complete network with one regime

In this section we consider a special case of a single-regime interbank network in which banks are homogeneous and all banks are connected to each other (*complete* network) with an equal level of interconnectedness. More precisely, we assume that $\theta_i = \bar{\theta}, \sigma_i = \bar{\sigma}, \eta_i = \bar{\eta}, \mu_i = \bar{\mu}, b_i = \bar{b}, q_i = \bar{q}, \delta_i = \bar{\delta}$ and $\pi_{i,j} = 1$ for all $i, j = 1, \dots, M$ and $j \neq i$. Furthermore, we will consider the case of a large network when the number of banks $M \to \infty$. Consequently, we set

$$
\zeta = \frac{\bar{\zeta}}{M-1} \qquad (21)
$$

for some constant $\bar{\zeta}$ so that the total level of interconnectedness of each bank, which is measured by the $\zeta \sum_{j \neq i} \pi_{i,j}$, remains the same at $\bar{\zeta}$, independent of the number of banks in the network.

The dynamic of the liquidity of bank i given by (8) is now simplified to

$$dX_i(t) = \alpha_i(t)dt + \bar{\zeta}[\bar{X}_{-i}(t) - X_i(t)]dt + \bar{\sigma}\sqrt{X_i(t)}dW_i(t) + \bar{\eta}X_i(t^-)dN(t) + \bar{\mu}[X_i(t) - \bar{\theta}]dt \quad (22)$$

where

$$\bar{X}_{-i}(t) = \frac{1}{M-1}\sum_{j \neq i} X_j(t) \quad (23)$$

denotes the average liquidity level of all banks except bank i, or the *peer's average* of bank i.

Although this simple case does not allow us to study the effect of reduction in the interconnectedness during a crisis regime, or the effect of panic-triggered liquidity outflows following a large shock, it will provide us some structural results for the optimal control policy when banks encounter idiosyncratic and systemic shocks, as well as a flight-to-quality effect. The results will be useful when we consider a more complex network where explicit solutions are not available.

4.1 Value functions

We first consider a system of M banks where $M < \infty$. We use $V_i(x)$ to denote the value function of bank i when the vector of liquidity levels is equal to x as we have only one regime. Under this homogeneous complete network assumption, the HJB equations reduce to

$$\min_{\alpha_i}\Bigg\{ \sum_{j=1}^{M} \frac{\partial V_i(x)}{\partial x_j}\left(\alpha_i \mathbf{1}_{j=i} + \hat{\alpha}_j(x)\mathbf{1}_{j\neq i} + \bar{\zeta}[\bar{x}_{-j} - x_j] + \bar{\mu}[x_j - \bar{\theta}]\right) + \frac{1}{2}\sum_{j=1}^{M}\frac{\partial^2 V_i(x)}{\partial x_j^2}\bar{\sigma}^2 x_j$$
$$+ \lambda\left[V_i(x + \bar{\eta}x) - V_i(x)\right] + \frac{1}{2}\alpha_i^2 - \bar{b}\alpha_i(\bar{\theta} - x_i) + \frac{1}{2}\bar{q}(x_i - \bar{\theta})^2 - \bar{\delta}V_i(x)\Bigg\} = 0 \quad (24)$$

where $\mathbf{1}_A$ is an indicator function taking value 1 if A is true and 0 otherwise, and $\hat{\alpha}_1(x), \ldots, \hat{\alpha}_M(x)$ simultaneously solve (24) for all $i = 1, \ldots, M$. This result is similar to the mean-field approach where banks compares its liquidity level with the network's average. Consequently, the value function of bank i can be defined as a function of its own liquidity x_i and the average of all other banks' liquidity $\bar{x}_{-i}$:

$$V_i(x) \equiv V_i(x_i, \bar{x}_{-i}) = \frac{1}{2}\gamma_1(x_i - \hat{\theta})^2 + \frac{1}{2}\gamma_2(\bar{x}_{-i} - x_i)^2 + \frac{1}{2}\gamma_3(\bar{x}_{-i} - \tilde{\theta})^2 + \gamma_4 \quad (25)$$

for some constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \hat{\theta}$ and $\tilde{\theta}$. Under the limiting case when the number of banks $M \to \infty$, we are able to derive their values.

Proposition 4.1. *Suppose that $\bar{q} > \bar{b}^2$. As $M \to \infty$, we have*

$$\gamma_1 = \sqrt{\left(\bar{\beta} + \frac{\bar{\zeta}}{2}\right)^2 + \bar{q} - \bar{b}^2} - \bar{\beta} - \frac{\bar{\zeta}}{2} \tag{26}$$

$$\gamma_2 = \sqrt{\left(\bar{\beta} + \bar{\zeta}\right)^2 + \bar{q} - \bar{b}^2} - \sqrt{\left(\bar{\beta} + \frac{\bar{\zeta}}{2}\right)^2 + \bar{q} - \bar{b}^2} - \frac{\bar{\zeta}}{2} \tag{27}$$

$$\gamma_3 = \frac{\bar{\zeta}\gamma_1}{2(\gamma_1 + \bar{\beta})} \tag{28}$$

$$\begin{aligned}\gamma_4 = \frac{1}{2\bar{\delta}}\Big\{ & -\gamma_1^2(\hat{\theta} - \bar{\theta})^2 - 2\gamma_1\gamma_3(\hat{\theta} - \bar{\theta})(\tilde{\theta} - \bar{\theta}) + (\gamma_1 + \gamma_2)\bar{\sigma}^2\bar{\theta} \\ & + \lambda\bar{\eta}\bar{\theta}\left[\gamma_1\big(\bar{\eta}\bar{\theta} - 2(\hat{\theta} - \bar{\theta})\big) + \gamma_3\big(\bar{\eta}\bar{\theta} - 2(\tilde{\theta} - \bar{\theta})\big)\right]\Big\} - \frac{\gamma_1}{2}(\hat{\theta} - \bar{\theta})^2 - \frac{\gamma_3}{2}(\tilde{\theta} - \bar{\theta})^2\end{aligned} \tag{29}$$

$$\hat{\theta} = \bar{\theta} + \frac{-\bar{\theta}\lambda\bar{\eta}(\bar{\eta} + 1) - \frac{1}{2}\bar{\sigma}^2\left(1 + \frac{\gamma_2}{\gamma_1}\right)}{\gamma_1 + \bar{\beta} + \bar{\zeta} + \frac{1}{2}\lambda\bar{\eta}^2 + \frac{\bar{\delta}}{2}} \tag{30}$$

$$\tilde{\theta} = \bar{\theta} + \frac{-\bar{\theta}\lambda\bar{\eta}(\bar{\eta} + 1) + (\gamma_1 + 2\bar{\beta})(\hat{\theta} - \bar{\theta})}{\gamma_1 + \bar{\beta} + \frac{1}{2}\lambda\bar{\eta}^2 + \frac{\bar{\delta}}{2}} \tag{31}$$

where

$$\bar{\beta} = \bar{b} - \bar{\mu} - \frac{1}{2}\lambda\bar{\eta}(\bar{\eta} + 2) + \frac{\bar{\delta}}{2}. \tag{32}$$

Proof. Substitute the value function (25) and its derivatives in the HJB equation (24) to obtain a system of equations. The results are obtained from solving the system of equations. □

So the minimum total discounted cost depends on three types of distances: (i) distance between x_i and $\hat{\theta}$, (ii) distance between $\bar{x}_{-i}$ and x_i, and (iii) distance between $\bar{x}_{-i}$ and $\tilde{\theta}$. From our assumption that $\bar{q} > \bar{b}^2$, we can see that $\gamma_1 > 0$ and it is not difficult to show that $\gamma_2 \leq 0$. If we assume further that $\bar{\beta} \geq 0$, which is true if $\bar{b} \geq \bar{\mu}$ and $\bar{\eta} \leq 0$, then $\gamma_3 \geq 0$. Interestingly, the first distance is the distance between the current liquidity x_i and an *adjusted* target $\hat{\theta}$ instead of its *original* target $\bar{\theta}$. This difference is related to the notion of *precautionary liquidity* that suggests banks to hold liquidity more or less than their original targets. Also, the fact that $\gamma_2 \leq 0$ is related to the notion of liquidity provision for the interbank relationship, in which banks that are net liquidity providers should raise their liquidity from outside the interbank network to increase the liquidity in the system. We discuss these two notions in more details in the next section. We finish this section with the following proposition.

Proposition 4.2. *Suppose that $\bar{q} > \bar{b}^2$. Then the following results hold:*

(i) $\gamma_1 > 0$

(ii) $\gamma_2 \leq 0$ *where* $\gamma_2 = 0$ *if, and only if,* $\bar{\zeta} = 0$.

(ii) If $\bar{\beta} \geq 0$ *then* $\gamma_3 \geq 0$ *where* $\gamma_3 = 0$ *if, and only if,* $\bar{\zeta} = 0$.

4.2 Optimal liquidity control

Based on the value function (25) and the optimal control structure (16), the optimal control for bank i at the equilibrium is given by

$$\hat{\alpha}_i(x_i, \bar{x}_{-i}) = \bar{b}(\bar{\theta} - x_i) + \gamma_1(\hat{\theta} - x_i) + \gamma_2(\bar{x}_{-i} - x_i). \tag{33}$$

The first term in (33) comes directly from the second term in the running cost (10), while the second and the third terms in (33) come directly from the value function (25). The first two terms suggest that the bank should control its liquidity toward some fixed targets. Combining these two terms, we can rewrite the optimal control (33) as follows:

$$\hat{\alpha}_i(x_i, \bar{x}_{-i}) = (\bar{b} + \gamma_1)\left([\bar{\theta} + \Delta\bar{\theta}] - x_i\right) + \gamma_2(\bar{x}_{-i} - x_i) \tag{34}$$

where

$$\Delta\bar{\theta} = [\hat{\theta} - \bar{\theta}]\left(\frac{\gamma_1}{\gamma_1 + \bar{b}}\right) = \frac{-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1)\gamma_1 - \frac{1}{2}\bar{\sigma}^2(\gamma_1 + \gamma_2)}{(\gamma_1 + \bar{b})(\gamma_1 + \bar{\beta} + \bar{\zeta} + \frac{1}{2}\lambda\bar{\eta}^2 + \frac{\bar{\delta}}{2})}. \tag{35}$$

That is, the bank should control its liquidity toward its original target $\bar{\theta}$ plus a *precautionary liquidity* $\Delta\bar{\theta}$ with a *total adjustment speed* of $\bar{b} + \gamma_1$. As mentioned earlier, the precautionary liquidity comes from the difference between the original target $\bar{\theta}$ and the adjusted target $\hat{\theta}$. It also depends on the ratio $\gamma_1/(\gamma_1 + \bar{b})$ which measures the relative importance between the future cost through γ_1 and the current running cost through $\bar{b}$. Without the future cost, $\gamma_1 = 0$ and it is optimal for the bank to drive its liquidity level toward its original target $\bar{\theta}$ at the speed of $\bar{b}$ to balance the controlling cost from the term $\frac{1}{2}\alpha_i^2$ and the reward from the term $-\bar{b}\alpha_i(\bar{\theta} - x_i)$ in the cost function (10). However, the bank has to consider the future cost and adjust its target to account for possible uncertainties such as idiosyncrasy from daily deposits and withdrawals, and systemic liquidity shocks that could cause its liquidity level to deviate from its optimal path. We refer to this new target $\bar{\theta} + \Delta\bar{\theta}$ as the *optimal target* $\bar{\theta}^*$. The bank also increases the adjustment speed from $\bar{b}$ to $\bar{b} + \gamma_1$ to bring its liquidity close to the optimal target faster, which will help reduce the cost bank has to pay in the future as long as its liquidity level is still not at the original target from the term $\frac{\bar{q}}{2}(\bar{\theta} - x_i)^2$ in (10).

To further understand the precautionary liquidity, let's consider the case when the relative size of the liquidity shock $\bar{\eta}$ is negative, and the incentive parameter $\bar{b}$ is greater than the flight-to-quality parameter $\bar{\mu}$. Under these assumptions, the expression in (35) implies that bank i should adjust its target upward due to the shock term $-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1)\gamma_1$ (positive precautionary) and it should adjust the value downward due to the volatility term $-\frac{1}{2}\bar{\sigma}^2(\gamma_1 + \gamma_2)$ (negative precautionary). Intuitively, as liquidity shocks are negative, the bank should hold extra liquidity so that once a shock occurs, its liquidity level would not be too far away from its original target. Observe further that $-\bar{\eta}(\bar{\eta}+1)$ increases as the shock size $\bar{\eta}$ goes from 0 down to -0.5 and it decreases as the shock size goes down from -0.5 to -1. This implies that bank should maintain higher liquidity as the relative shock size becomes larger in magnitude upto the point where the shock size is equal to -50%. This is to make sure that when a shock occurs, the liquidity level of the bank will not be too low relatively to its original target $\bar{\theta}$. However, if the shock size gets larger than -50% in magnitude, the bank should not further increase the level of its target, but it should hold less precautionary liquidity compared to that when the shock size is -50%. The reason is that there is no benefit of keeping the liquidity level too high if a jump will wipe out most of it.

On the other hand, the volatility term suggests that bank should maintain its liquidity lower than its original target $\bar{\theta}$. This term comes from the fact that the volatility of the liquidity depends on the liquidity level through the term $\sqrt{X_i(t)}$. So banks have an incentive to keep the liquidity lower than their original targets $\bar{\theta}$ to reduce the volatility. This adjustment term due to volatility will disappear if the volatility term in (8) does not depend on $X(t)$.

The last term in (34) suggests the bank to hold *liquidity provision for interbank relationship*. More specifically, assuming $\bar{\zeta} > 0$, when the average liquidity in the network is lower than the bank's liquidity (equivalently $\bar{x}_{-i} < x_i$), the interbank relationship given in (22) demands bank i to be a net liquidity provider. To server this role, it is optimal for bank i to raise its liquidity from outside the network from the term $\gamma_2(\bar{x}_{-i} - x_i)$ in (34) which is positive. On the other hand, when the average liquidity in the network is higher than bank i's liquidity (equivalently $\bar{x}_{-i} > x_i$), the term $\gamma_2(\bar{x}_{-i} - x_i)$ is negative, implying that bank i should hold negative liquidity provision to reduce the liquidity in the system and at the same time act as a net borrower in the interbank network as demanded from the interbank relationship in (22). The size of the liquidity provision depends on how much bank i is expected to lend to/borrow from its peer banks $(\bar{x}_{-i} - x_i)$ and the coefficient γ_2. We refer to γ_2 as the *proportional liquidity provision.* In this section in which all banks are homogeneous, we simply call it *liquidity provision.*

Now observe from (22) that the total level of interconnectedness of each bank is equal to $\bar{\zeta}$. When banks hold liquidity provision for the interbank relationship, the total level of the interconnectedness reduces to

$$\bar{\zeta} + \gamma_2 = \frac{\bar{\zeta}}{2} + \sqrt{(\bar{\beta} + \bar{\zeta})^2 + \bar{q} - \bar{b}^2} - \sqrt{\left(\bar{\beta} + \frac{\bar{\zeta}}{2}\right)^2 + \bar{q} - \bar{b}^2} > 0. \tag{36}$$

It is easy to see that $\bar{\zeta}/2 \leq \bar{\zeta} + \gamma_2 \leq \bar{\zeta}$. That is, banks choose to use liquidity provision to reduce the total level of interconnectedness by no more than 50%. Recall that under the cost function (10), banks try to keep their liquidity levels close to their targets as much as they can, and at the same time they try not to control their liquidity too much to keep the cost of adding or removing liquidity by borrowing from or lending to a party outside the interbank system as small as possible. The role of the liquidity provision for the interbank relationship is to reduce the impact from the interbank relationship on the liquidity level so that banks can control their liquidity levels more effectively. We now consider the effects of the interconnectedness levels, systemic liquidity shocks and the flight-to-quality flows on the precautionary liquidity and liquidity provision terms.

4.3 Effect of interconnectedness level

As the interconnectedness level $\bar{\zeta}$ increases, the impact from the interbank relationship increases. One might think that it is optimal for banks to hold more liquidity provision to reduce this impact, and increase the adjustment speed toward the optimal target to ensure that the liquidity levels are not too far from the targets. However, we show that it is optimal for banks to reduce the adjustment speed $\bar{b}+\gamma_1$, and when the interconnectedness level is sufficiently high, it is optimal for banks to hold less liquidity provision as well.

Proposition 4.3. *Suppose $\bar{q} > \bar{b}^2$. Then the following results hold:*

(i) γ_1 is decreasing in $\bar{\zeta}$.

(ii) There exists $\zeta^* > 0$ such that γ_2 is decreasing in $\bar{\zeta}$ on $[0, \zeta^*)$ and it is increasing in $\bar{\zeta}$ on (ζ^*, ∞).

(iii) Suppose $\bar{\eta} < 0$ and $\bar{b} > \bar{\mu}$.

(a) If $-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) < \frac{\bar{\sigma}^2}{4}$, then $\Delta\bar{\theta} < 0$ and is increasing in $\bar{\zeta}$.

(b) If $-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) \geq \bar{\sigma}^2$, then $\Delta\bar{\theta} > 0$ and is decreasing in $\bar{\zeta}$.

(iv) The limits of γ_1, γ_2 and $\Delta\bar{\theta}$ as $\bar{\zeta} \to \infty$ are zero.

Proof. Consider

$$\frac{\partial \gamma_1}{\partial \bar{\zeta}} = \frac{1}{2}\left[\frac{\bar{\beta}+\frac{\bar{\zeta}}{2}}{\sqrt{\left(\bar{\beta}+\frac{\bar{\zeta}}{2}\right)^2+\bar{q}-\bar{b}^2}} - 1\right] < 0, \tag{37}$$

which proves (i). Now consider

$$\frac{\partial \gamma_2}{\partial \bar{\zeta}} = \frac{1}{2}\left[\frac{\bar{\beta}+\bar{\zeta}}{\sqrt{(\bar{\beta}+\bar{\zeta})^2+\bar{q}-\bar{b}^2}} - \frac{\bar{\beta}+\frac{\bar{\zeta}}{2}}{\sqrt{(\bar{\beta}+\frac{\bar{\zeta}}{2})^2+\bar{q}-\bar{b}^2}}\right] - \frac{1}{2}\left[1 - \frac{\bar{\beta}+\bar{\zeta}}{\sqrt{(\bar{\beta}+\bar{\zeta})^2+\bar{q}-\bar{b}^2}}\right]. \tag{38}$$

It can be seen that $\frac{\partial \gamma_2}{\partial \bar{\zeta}} < 0$ when $\bar{\zeta} = 0$, and that $\frac{\partial \gamma_2}{\partial \bar{\zeta}} \to 0$ as $\bar{\zeta} \to \infty$. Now consider the second order derivative:

$$\frac{\partial^2 \gamma_2}{\partial \bar{\zeta}^2} = \frac{\bar{q}-\bar{b}^2}{\left[(\bar{\beta}+\bar{\zeta})^2+\bar{q}-\bar{b}^2\right]^{3/2}} - \frac{\bar{q}-\bar{b}^2}{4\left[(\bar{\beta}+\frac{\bar{\zeta}}{2})^2+\bar{q}-\bar{b}^2\right]^{3/2}}. \tag{39}$$

It is not difficult to show that the equation $\frac{\partial^2\gamma_2}{\partial\bar{\zeta}^2} = 0$ has one positive ($\bar{\zeta}_+ > 0$) and one negative ($\bar{\zeta}_- < 0$) real roots, and that $\frac{\partial^2\gamma_2}{\partial\bar{\zeta}^2} > 0$ when $\bar{\zeta} = 0$ and $\frac{\partial^2\gamma_2}{\partial\bar{\zeta}^2} < 0$ for sufficiently large $\bar{\zeta}$. So by the continuity of $\frac{\partial^2\gamma_2}{\partial\bar{\zeta}^2}$, we have $\frac{\partial^2\gamma_2}{\partial\bar{\zeta}^2} > 0$ for $\bar{\zeta} \in [0, \bar{\zeta}_+)$ and $\frac{\partial^2\gamma_2}{\partial\bar{\zeta}^2} < 0$ for $\bar{\zeta} \in (\bar{\zeta}_+, \infty)$. As a consequent, $\frac{\partial\gamma_2}{\partial\bar{\zeta}}$ is increasing in $\bar{\zeta}$ on $[0, \bar{\zeta}_+)$ and decreasing in $\bar{\zeta}$ on $(\bar{\zeta}_+, \infty)$. Since $\frac{\partial\gamma_2}{\partial\bar{\zeta}} < 0$ when $\bar{\zeta} = 0$, and that $\frac{\partial\gamma_2}{\partial\bar{\zeta}} \to 0$ as $\bar{\zeta} \to \infty$, the continuity of $\frac{\partial\gamma_2}{\partial\bar{\zeta}}$ implies that $\frac{\partial\gamma_2}{\partial\bar{\zeta}} = 0$ at some $\zeta^* \in (0, \bar{\zeta}_+)$. Consequently, the value of γ_2 must be decreasing for $\bar{\zeta} \in [0, \bar{\zeta}^*)$ and increasing for $\bar{\zeta} \in (\bar{\zeta}^*, \infty)$. This completes the proof for (ii).

It can be shown that

$$\frac{\partial\Delta\bar{\theta}}{\partial\bar{\zeta}} = \gamma_1 \left[A \left(\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) + \frac{\bar{\sigma}^2}{2} \left(\frac{\gamma_1+\gamma_2}{\gamma_1} \right) B \right) + C \left(\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) + \frac{\bar{\sigma}^2}{2} \left(\frac{\gamma_1+\gamma_2}{\gamma_1} \right) \right) \right] / D \quad (40)$$

where

$$A = \frac{1}{2} \left(\frac{\gamma_1 + \bar{b}}{\gamma_1 + \bar{\beta} + \frac{\bar{\zeta}}{2}} \right) \left(\gamma_1 + \bar{\zeta} + \bar{\beta} + \frac{1}{2}\lambda\bar{\eta}^2 + \frac{\bar{\delta}}{2} \right) \quad (41)$$

$$B = 2 \left(\frac{\gamma_1 + \bar{\beta} + \frac{\bar{\zeta}}{2}}{\gamma_1 + \gamma_2 + \bar{\beta} + \bar{\zeta}} \right) \quad (42)$$

$$C = \bar{b} - \frac{1}{2} \left(\bar{\mu} + \lambda\bar{\eta}(\bar{\eta}+1) \right) \left(\frac{\gamma_1}{\gamma_1 + \bar{\beta} + \frac{\bar{\zeta}}{2}} \right) \quad (43)$$

$$D = (\gamma_1 + \bar{b})^2 \left(\gamma_1 + \bar{\beta} + \bar{\zeta} + \frac{1}{2}\lambda\bar{\eta}^2 + \frac{\bar{\delta}}{2} \right)^2. \quad (44)$$

It is easy to see that $A > 0$, $D > 0$ and under the assumption that $\bar{\eta} < 0$ and $\bar{b} > \bar{\mu}$, $C > 0$. It is not difficult to show that

$$\frac{\gamma_1 + \bar{\beta} + \frac{\bar{\zeta}}{2}}{\gamma_1 + \gamma_2 + \bar{\beta} + \bar{\zeta}} = \sqrt{\frac{\left(\bar{\beta} + \frac{\bar{\zeta}}{2}\right)^2 + \bar{q} - \bar{b}^2}{\left(\bar{\beta} + \bar{\zeta}\right)^2 + \bar{q} - \bar{b}^2}} \quad (45)$$

is equal to 1 when $\bar{\zeta} = 0$ and it is decreasing in $\bar{\zeta}$ and converges to $\frac{1}{2}$ as $\bar{\zeta}$ approaches infinity when $\bar{\beta} > 0$ which is true if $\bar{\eta} < 0$ and $\bar{b} > \bar{\mu}$. Now $\frac{\gamma_1+\gamma_2}{\gamma_1}$ can be written as

$$\frac{\gamma_1 + \gamma_2}{\gamma_1} = \frac{\sqrt{\left(\bar{\beta} + \frac{\bar{\zeta}}{2}\right)^2 + \bar{q} - \bar{b}^2} + \bar{\beta} + \frac{\bar{\zeta}}{2}}{\sqrt{(\bar{\beta} + \bar{\zeta})^2 + \bar{q} - \bar{b}^2} + \bar{\beta} + \bar{\zeta}} \quad (46)$$

which is equal to 1 when $\bar{\zeta} = 0$, decreasing in $\bar{\zeta}$, and converges to $\frac{1}{2}$ as $\bar{\zeta} \to \infty$. As a consequence, we have $\frac{\gamma_1+\gamma_2}{\gamma_1} \in (\frac{1}{2}, 1]$ and $\frac{\gamma_1+\gamma_2}{\gamma_1} B \in (\frac{1}{2}, 2]$.

Now let's consider the case when $-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) < \frac{\bar{\sigma}^2}{4}$. We have

$$\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) + \frac{\bar{\sigma}^2}{2}\left(\frac{\gamma_1+\gamma_2}{\gamma_1}\right)B > -\frac{\bar{\sigma}^2}{4} + \frac{\bar{\sigma}^2}{4} = 0 \tag{47}$$

$$\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) + \frac{\bar{\sigma}^2}{2}\left(\frac{\gamma_1+\gamma_2}{\gamma_1}\right) > -\frac{\bar{\sigma}^2}{4} + \frac{\bar{\sigma}^2}{4} = 0 \tag{48}$$

and thus from (35) and (48), we have $\Delta\bar{\theta} < 0$ and from (40), (47) and (48), we have $\frac{\partial\Delta\bar{\theta}}{\partial\bar{\zeta}} > 0$. For $-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) \geq \bar{\sigma}^2$, we have

$$\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) + \frac{\bar{\sigma}^2}{2}\left(\frac{\gamma_1+\gamma_2}{\gamma_1}\right)B \leq -\bar{\sigma}^2 + \bar{\sigma}^2 = 0 \tag{49}$$

$$\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) + \frac{\bar{\sigma}^2}{2}\left(\frac{\gamma_1+\gamma_2}{\gamma_1}\right) \leq -\bar{\sigma}^2 + \frac{\bar{\sigma}^2}{2} < 0 \tag{50}$$

and thus from (35) and (50) we have $\Delta\bar{\theta} > 0$ and from (40), (49) and (50), we have $\frac{\partial\Delta\bar{\theta}}{\partial\bar{\zeta}} < 0$. This completes the proof for (iii). To prove that γ_1 and γ_2 converge to zero as $\bar{\zeta} \to \infty$, we use the fact that $\lim_{x\to\infty}\sqrt{x^2+a} - x = 0$ for any positive constant a. Then $\Delta\bar{\theta}$ converges to zero as γ_1 and γ_2 converge to zero. □

From the above results, we can see that, when the interconnectedness level is sufficiently small ($\bar{\zeta} < \zeta^*$), it is optimal for banks to increase liquidity provision for the interbank relationship as the level of interconnectedness increases to reduce the impact of the interbank relationship, and at the same time banks should reduce the speed of adjusting their liquidity toward their optimal targets as the benefit of controlling their own liquidity can be less if it is going against a stronger interbank relationship. Once the interconnectedness level is large enough ($\bar{\zeta} > \zeta^*$), the benefit of reducing the interbank impact by holding liquidity provision is no longer worth the increased cost. So it is optimal for banks to lower their liquidity provision for the interbank relationship.

Now we focus on the effect of the interconnectedness level on the precautionary liquidity for negative shocks. When the effect from the liquidity shocks is sufficiently less than the volatility effect ($-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) < \frac{\bar{\sigma}^2}{4}$), banks always hold negative precautionary liquidity. However, when the interconnectedness level increases, the banks' liquidity levels are moving toward the peer average with faster speed, and thus the effect from unexpected changes in the liquidity level due to the volatility term quickly disappears by the interbank relationship. As a result, the benefit from keeping the liquidity lower than the original target to reduce the volatility is less. So banks should hold less negative precautionary liquidity. On the other hand, when the effect from the liquidity shocks is large enough compared to the volatility effect ($-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) \geq \bar{\sigma}^2$), banks always hold positive precautionary liquidity. However, when the interconnectedness level increases, a bigger portion of the bank's positive precautionary liquidity is likely to be shared to the peer banks, and hence its benefit is less. As a consequence, banks hold less precautionary liquidity. When the effect from the liquidity shocks and that from the volatility are not much different ($\frac{\bar{\sigma}^2}{4} \le -\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) < \bar{\sigma}^2$), the relationship between the precautionary and the level of interconnectedness is unclear. However, it is worth noting that the importance of the effect of the liquidity shocks depends also on γ_1, and the importance of the the effect of the volatility depends also on $\gamma_1 + \gamma_2$ as can be seen from (35). When the interconnectedness level is low ($\bar{\zeta} < \zeta^*$), γ_1 and γ_2 decrease as the interconnectedness level increases. So the importance of the effect from the volatility is reduced more, and it is possible that banks hold more precautionary liquidity as the liquidity shocks become relatively more important. Nevertheless, when the level of interconnectedness is large enough ($\bar{\zeta} > \zeta^*$), γ_2 becomes increasing in the interconnectedness level, making the relative importance of the volatility effect higher. As a result, banks could now reduce the precautionary liquidity as the interconnectedness level increases. We, in fact, see this behavior in our numerical examples presented below. As the level of interconnectedness level approaches infinity, all of γ_1, γ_2 and $\Delta\bar{\theta}$ converge to zero, implying banks hold zero liquidity provision and precautionary liquidity when the interbank relationship is very strong. That is, there is no benefit to control liquidity for their own future as any benefit will be quickly shared to their peer banks. The optimal control for the banks in that case is to focus on the current reward as determined by the incentive parameter $\bar{b}$.

4.4 Effects of systemic liquidity shocks

There are two key parameters for systemic liquidity shocks. The first is the shock arrival rate $\bar{\lambda}$, and second is the relative size of the shock $\bar{\eta}$. In this section we focus on the negative shocks $\bar{\eta} \in (-1, 0)$ to represent adverse shocks such as bank runs. Also, we assume that $\bar{\zeta} > 0$ when we analyze the effect of γ_2 since $\gamma_2 = 0$ if $\bar{\zeta} = 0$. We have the following results.

Proposition 4.4. *Suppose $\bar{q} < \bar{b}^2$ and $\bar{\eta} \in (-1, 0)$. Then γ_1 is decreasing in the arrival rate (λ) and the absolute shock size ($|\bar{\eta}|$), while γ_2 is increasing in the arrival rate (λ) and the absolute shock size ($|\bar{\eta}|$) when $\bar{\zeta} > 0$. In addition, $\gamma_1 \downarrow 0$, $\gamma_2 \uparrow 0$ and $\Delta\bar{\theta} \to 0$ as $\lambda \to \infty$.*

Proof. It can be shown that

$$\frac{\partial \gamma_1}{\partial \lambda} = \frac{\bar{\eta}}{2}(\bar{\eta}+2)\left[1 - \frac{\bar{\beta} + \frac{\bar{\zeta}}{2}}{\sqrt{\left(\bar{\beta} + \frac{\bar{\zeta}}{2}\right)^2 + \bar{q} - \bar{b}^2}}\right] < 0, \tag{51}$$

and that

$$\frac{\partial \gamma_2}{\partial \lambda} = -\frac{\bar{\eta}}{2}(\bar{\eta}+2)\left[\frac{\bar{\beta} + \bar{\zeta}}{\sqrt{(\bar{\beta} + \bar{\zeta})^2 + \bar{q} - \bar{b}^2}} - \frac{\bar{\beta} + \frac{\bar{\zeta}}{2}}{\sqrt{\left(\bar{\beta} + \frac{\bar{\zeta}}{2}\right)^2 + \bar{q} - \bar{b}^2}}\right] > 0 \tag{52}$$

if $\bar{\zeta} > 0$. For $\bar{\eta} \in (-1, 0)$, it can be shown that

$$\frac{\partial \gamma_1}{\partial \bar{\eta}} = \lambda(\bar{\eta}+1) \left[1 - \frac{\bar{\beta} + \frac{\bar{\zeta}}{2}}{\sqrt{\left(\bar{\beta} + \frac{\bar{\zeta}}{2}\right)^2 + \bar{q} - \bar{b}^2}} \right] > 0, \tag{53}$$

and that

$$\frac{\partial \gamma_2}{\partial \bar{\eta}} = -\lambda(\bar{\eta}+1) \left[\frac{\bar{\beta} + \bar{\zeta}}{\sqrt{\left(\bar{\beta} + \bar{\zeta}\right)^2 + \bar{q} - \bar{b}^2}} - \frac{\bar{\beta} + \frac{\bar{\zeta}}{2}}{\sqrt{\left(\bar{\beta} + \frac{\bar{\zeta}}{2}\right)^2 + \bar{q} - \bar{b}^2}} \right] < 0 \tag{54}$$

if $\bar{\zeta} > 0$. To prove the limit of γ_1 and γ_2, note that $\bar{\beta} \to \infty$ as $\lambda \to \infty$ for $\bar{\eta} < 0$, and the result follows from the fact that $\lim_{x \to \infty} \sqrt{x^2 + a} - x = 0$ for any positive constant a. The limit of $\Delta\bar{\theta}$ is clear from (35) and the limits of γ_1 and γ_2. This completes the proof. □

The above results imply that when the systemic liquidity shocks become more significant either through a higher arrival rate or a larger negative shock size, it is optimal for banks to hold less liquidity provision for the interbank relationship and to lower the adjustment speed toward the optimal target. This simply comes from the fact that higher arrival rate of negative shocks or larger negative shock sizes will reduce the liquidity banks prepare for their future, and thus the benefit of having liquidity provision or moving faster toward the optimal target is greatly reduced. In other words, banks choose to accept the consequences of worsen adverse shocks rather than fighting against them because the cost of doing so is too expensive.

The relationship between the precautionary liquidity $\Delta\bar{\theta}$ and the shock's characteristics λ and $\bar{\eta}$ is complex and unclear. However, as discussed in Section 4.2, the effect of the liquidity shocks $-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1)$ is clearly increasing in λ and is large for shocks of size around -50%. The effect is small for small absolute shock sizes (low impact) or very large absolute shock sizes (no value for precautionary liquidity as it will be mostly gone when shocks occur). However, as the arrival rate or the absolute size of the negative shock increases, the importance of the effect of the shocks on the precautionary liquidity as measured by γ_1 decreases. In fact, as the shock arrival rate goes to infinity, banks hold zero liquidity provision for the interbank relationship and zero precautionary liquidity with no additional adjustment speed. This behavior of banks may induce an increased systemic risk of the banking system.

4.5 Effects of flight-to-quality flows

The degree of flight-to-quality effect is measured by $\bar{\mu}$. The higher the value of $\bar{\mu}$, the faster the liquidity flows out of low quality banks and the faster the liquidity flows into high quality banks. We have the following results.

Proposition 4.5. *Suppose $\bar{q} > \bar{b}^2$. Then γ_1 is increasing in $\bar{\mu}$, and γ_2 is decreasing in $\bar{\mu}$ when $\bar{\zeta} > 0$.*

Proof. It can be shown that

$$\frac{\partial \gamma_1}{\partial \bar{\mu}} = 1 - \frac{\bar{\beta} + \frac{\bar{\zeta}}{2}}{\sqrt{\left(\bar{\beta} + \frac{\bar{\zeta}}{2}\right)^2 + \bar{q} - \bar{b}^2}} > 0, \tag{55}$$

and that

$$\frac{\partial \gamma_2}{\partial \bar{\mu}} = -\left[\frac{\bar{\beta} + \bar{\zeta}}{\sqrt{\left(\bar{\beta} + \bar{\zeta}\right)^2 + \bar{q} - \bar{b}^2}} - \frac{\bar{\beta} + \frac{\bar{\zeta}}{2}}{\sqrt{\left(\bar{\beta} + \frac{\bar{\zeta}}{2}\right)^2 + \bar{q} - \bar{b}^2}}\right] < 0 \tag{56}$$

if $\bar{\zeta} > 0$. □

The above results imply that when the flight-to-quality effect is stronger, banks hold liquidity provision for the interbank relationship more aggressively to further reduce the impact from the interbank relationship. They also increase the adjustment speed for their liquidity toward their optimal targets. These actions are opposite to the case in which the arrival rate or the size of the negative liquidity shock increases. There are three main differences between the liquidity shocks and the flight-to-quality flows in our model. First, shocks are assumed to be negative, while the flight-to-quality flows have both negative and positive effects. That is, banks with liquidity levels being below their targets encounter liquidity outflows, but banks with liquidity levels being above their targets receive liquidity inflows. So keeping the liquidity level high not only avoids the liquidity outflows, but the banks also receive liquidity inflows. Second, the effect of the liquidity shocks is larger when the liquidity levels of the banks are high. This is because the size of the shock is a fixed percentage of the current liquidity level or $\bar{\eta}X_i(t)$ for a fixed $\bar{\eta}$. On the contrary, the flight-to-quality effect is less if the liquidity levels of the banks are close to their targets. So banks have incentive to respond to the flight-to-quality effect by controlling their liquidity levels to be close or even above their targets. Third, the effect of liquidity shocks tends to have less impact for consequent shocks, but the flight-to-quality flows tend to get amplified over time. More specifically, an arrival of a shock reduces the banks' liquidity, and hence the absolute impact from the next shock is likely to be smaller. On the other hand, when a bank encounters flight-to-quality outflows, its liquidity level tends to go further below its target, and as a consequence makes the flight-to-quality outflows larger due to a larger gap between the liquidity level and the target. As the results of Proposition 4.5 show, the benefit for banks to respond to an increased effect of the flight-to-quality flows more aggressively outweighs the cost.

The effect of flight-to-quality flows on the precautionary liquidity level is complex. However, from the numerator of $\Delta\bar{\theta}$ in (35), the importance of the liquidity shock term $-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1)$ is measured by γ_1, which is increasing in the degree of flight-to-quality effect $\bar{\mu}$. It can also be shown that $\gamma_1+\gamma_2$, which measures the importance of the volatility term $\frac{1}{2}\bar{\sigma}^2$, is also increasing in $\bar{\mu}$ but at a slower rate. Hence, we expect that, when either the shock or volatility term is sufficiently larger than the other term, banks tend to increase the magnitude of their precautionary liquidity levels when the flight-to-quality effect is stronger. That is, banks reinforce their precautionary liquidity policy more in response to a higher flight-to-quality effect. We in fact observe this behavior in our numerical examples below.

4.6 Effects of reward and penalty parameters

There are two parameters in the cost function (10). The first parameter is the incentive or reward parameter $\bar{b}$. This parameter provides a reward to any bank that controls its liquidity level in the direction that makes it closer to the target. This could represent reductions in the lending rates at the central bank's lending facilities. The second parameter is the penalty parameter $\bar{q}$ which penalizes any bank that has liquidity level above or below its target. This could represent a loss in the bank's profit due to having a sub-optimal liquidity level. However, the central bank may impose additional fees on banks that hold liquidity above or below their targets to reinforce the balance between the stability/resilience of the banking system and its efficiency. Hence, the levels of $\bar{b}$ and $\bar{q}$ can be controlled partly or entirely by the central bank. We study their impacts on the banks' optimal policy and discuss some policy implications. We first present the results.

Proposition 4.6. *Suppose $\bar{q} > \bar{b}^2$. Then γ_1 is decreasing in $\bar{b}$ and increasing in $\bar{q}$, while γ_2 is increasing in $\bar{b}$ and decreasing in $\bar{q}$ when $\bar{\zeta} > 0$. Assume further that $\bar{\beta} > \max(0, \bar{b} - \frac{\bar{\zeta}}{2})$. Then the following results hold.*

(i) $\gamma_1 + \bar{b}$ *is increasing in* $\bar{b}$.

(ii) If $-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) > \frac{\bar{\sigma}^2}{2}$ *then* $\Delta\bar{\theta}$ *is positive and is decreasing in* $\bar{b}$.

(iii) If $-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) < \frac{\bar{\sigma}^2}{8}$ *then* $\Delta\bar{\theta}$ *is negative and is increasing in* $\bar{b}$.

(iv) γ_1, γ_2 *and* $\Delta\bar{\theta}$ *converge to zero as* $\bar{b} \uparrow \sqrt{\bar{q}}$.

(v) As $\bar{q} \uparrow \infty$, *we have* $\gamma_1 \uparrow \infty, \gamma_2 \downarrow -\frac{\bar{\zeta}}{2}$ *and* $\Delta\bar{\theta} \to 0$.

Proof. It can be shown that

$$\frac{\partial \gamma_1}{\partial \bar{b}} = \frac{\bar{\beta} + \frac{\bar{\zeta}}{2} - \bar{b}}{\sqrt{\left(\bar{\beta} + \frac{\bar{\zeta}}{2}\right)^2 + \bar{q} - \bar{b}^2}} - 1 < 0, \tag{57}$$

and that

$$\frac{\partial \gamma_2}{\partial \bar{b}} = \frac{\bar{\beta} + \bar{\zeta} - \bar{b}}{\sqrt{\left(\bar{\beta} + \bar{\zeta}\right)^2 + \bar{q} - \bar{b}^2}} - \frac{\bar{\beta} + \frac{\bar{\zeta}}{2} - \bar{b}}{\sqrt{\left(\bar{\beta} + \frac{\bar{\zeta}}{2}\right)^2 + \bar{q} - \bar{b}^2}} > 0 \tag{58}$$

if $\bar{\zeta} > 0$. Also, it is clear that γ_1 is increasing in $\bar{q}$. For γ_2, it can be shown that

$$\frac{\partial \gamma_2}{\partial \bar{q}} = \frac{1}{2}\left[\frac{1}{\sqrt{\left(\bar{\beta}+\bar{\zeta}\right)^2+\bar{q}-\bar{b}^2}} - \frac{1}{\sqrt{\left(\bar{\beta}+\frac{\bar{\zeta}}{2}\right)^2+\bar{q}-\bar{b}^2}}\right] < 0 \tag{59}$$

if $\bar{\zeta} > 0$.

Now let's consider the precautionary liquidity $\Delta\bar{\theta}$. It can be shown that

$$\frac{\partial \Delta\bar{\theta}}{\partial \bar{b}} = \left[A\left(\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1)+\frac{\bar{\sigma}^2}{2}B\right)+C\left(\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1)+\frac{\bar{\sigma}^2}{2}\left(\frac{\gamma_1+\gamma_2}{\gamma_1}\right)\right)\right]/D \tag{60}$$

where

$$A = \left(\frac{(\gamma_1+\bar{b})^2}{\gamma_1+\bar{\beta}+\frac{\bar{\zeta}}{2}}\right)\left(\gamma_1+\bar{\zeta}+\bar{\beta}+\frac{1}{2}\lambda\bar{\eta}^2+\frac{\bar{\delta}}{2}\right) \tag{61}$$

$$B = \left(\frac{\gamma_1+\bar{\beta}+\frac{\bar{\zeta}}{2}}{\gamma_1+\gamma_2+\bar{\beta}+\bar{\zeta}}\right)\left(\frac{\gamma_1+\gamma_2+\bar{b}}{\gamma_1+\bar{b}}\right) \tag{62}$$

$$C = \gamma_1\left(2\gamma_1+\bar{\zeta}+\bar{\beta}+\bar{b}+\frac{1}{2}\lambda\bar{\eta}^2+\frac{\bar{\delta}}{2}\right)\left(\frac{\bar{\beta}+\frac{\bar{\zeta}}{2}-\bar{b}}{\gamma_1+\bar{\beta}+\frac{\bar{\zeta}}{2}}\right) \tag{63}$$

$$D = (\gamma_1+\bar{b})^2\left(\gamma_1+\bar{\beta}+\bar{\zeta}+\frac{1}{2}\lambda\bar{\eta}^2+\frac{\bar{\delta}}{2}\right)^2. \tag{64}$$

It is clear that $D > 0$. Now if $\bar{\beta} > \max(0, \bar{b}-\frac{\bar{\zeta}}{2})$, then it can be shown that $A > 0$ and $C > 0$. From the proof of Proposition 4.3 we know that

$$\frac{\gamma_1+\bar{\beta}+\frac{\bar{\zeta}}{2}}{\gamma_1+\gamma_2+\bar{\beta}+\bar{\zeta}} = \sqrt{\frac{\left(\bar{\beta}+\frac{\bar{\zeta}}{2}\right)^2+\bar{q}-\bar{b}^2}{\left(\bar{\beta}+\bar{\zeta}\right)^2+\bar{q}-\bar{b}^2}} \tag{65}$$

is equal to 1 when $\bar{\zeta} = 0$ and it is decreasing in $\bar{\zeta}$ and converges to $\frac{1}{2}$ as $\bar{\zeta}$ approaches infinity when $\bar{\beta} > 0$ which is true if $\bar{\mu} < \min(\bar{b}, \frac{\bar{\zeta}}{2}) + \frac{\bar{\delta}}{2} - \frac{1}{2}\lambda\bar{\eta}(\bar{\eta}+2)$. We also know that $\frac{\gamma_1+\gamma_2}{\gamma_1} \in (\frac{1}{2}, 1]$. Now observe that

$$\frac{\gamma_1+\gamma_2+\bar{b}}{\gamma_1+\bar{b}} = \frac{\left(\frac{\gamma_1+\gamma_2}{\gamma_1}\right)\gamma_1+\bar{b}}{\gamma_1+\bar{b}} > \frac{\frac{1}{2}\gamma_1+\bar{b}}{\gamma_1+\bar{b}} = \frac{1}{2}+\frac{1}{2}\left(\frac{\bar{b}}{\gamma_1+\bar{b}}\right) \geq \frac{1}{2} \tag{66}$$

and that it is less than or equal to one as $\gamma_2 \leq 0$. Combining these results, we have $B \in (\frac{1}{4}, 1]$.

Now let's consider the case in which $-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) > \frac{\bar{\sigma}^2}{2}$. Then we have

$$\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1)+\frac{\bar{\sigma}^2}{2}B < -\frac{\bar{\sigma}^2}{2}+\frac{\bar{\sigma}^2}{2} = 0 \tag{67}$$

$$\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1)+\frac{\bar{\sigma}^2}{2}\left(\frac{\gamma_1+\gamma_2}{\gamma_1}\right) < -\frac{\bar{\sigma}^2}{2}+\frac{\bar{\sigma}^2}{2} = 0. \tag{68}$$

So from (35) and (68), we have $\Delta\bar{\theta} > 0$, and from (60), (67) and (68), we have $\frac{\partial \Delta\bar{\theta}}{\partial \bar{b}} < 0$.

Now consider the case in which $-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) < \frac{\bar{\sigma}^2}{8}$. Then we have

$$\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) + \frac{\bar{\sigma}^2}{2}B > -\frac{\bar{\sigma}^2}{8} + \frac{\bar{\sigma}^2}{8} = 0 \tag{69}$$

$$\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) + \frac{\bar{\sigma}^2}{2}\left(\frac{\gamma_1+\gamma_2}{\gamma_1}\right) > -\frac{\bar{\sigma}^2}{8} + \frac{\bar{\sigma}^2}{4} = \frac{\bar{\sigma}^2}{8} > 0. \tag{70}$$

So from (35) and (70), we have $\Delta\bar{\theta} < 0$, and from (60), (69) and (70), we have $\frac{\partial \Delta\bar{\theta}}{\partial \bar{b}} > 0$. Finally, the proof for (iv) and (v) is straightforward. □

Recall that banks adjust their liquidity levels toward their optimal targets with speed $\bar{b} + \gamma_1$ where $\bar{b}$ comes from the incentive or reward term in the cost function, and γ_1 comes from a component in the value function. Banks use γ_1 to increase the speed above $\bar{b}$ to minimize the cost in the future. From the above results, we can see that when the incentive parameter $\bar{b}$ increases, banks increase the total adjustment speed but lower the γ_1 component in response to the increase in the $\bar{b}$ component. Banks also reduce their liquidity provision as now they can obtain the liquidity outside the interbank network at a lower cost (higher reward). When the liquidity shock term is sufficiently higher than the volatility term in the precautionary liquidity ($-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) > \frac{\bar{\sigma}^2}{2}$), banks hold positive precautionary liquidity, but they decrease it as the incentive parameter $\bar{b}$ increases. On the other hand, when the volatility term significantly dominates the liquidity shock term ($-\bar{\theta}\lambda\bar{\eta}(\bar{\eta}+1) < \frac{\bar{\sigma}^2}{8}$), banks hold negative precautionary liquidity, and they increase it as $\bar{b}$ becomes higher. That is, banks reduce the magnitude of the precautionary liquidity level knowing that when more liquidity is needed, it can be obtained at a low cost. That also means the benefit of keeping the volatility low with negative precautionary liquidity is less. As the results show, when $\bar{b}$ is closing to its upper limit $\sqrt{\bar{q}}$, banks simply drive their liquidity toward their original target $\bar{\theta}$ at the speed of $\bar{b}$ without holding any liquidity provision or precautionary liquidity. If $\bar{b}$ is large enough, this implies that banks would respond quickly to negative shocks or any type of liquidity drop by injecting more liquidity into the system utilizing the central bank's lending facilities. As a result, the overall stability of the banking system may be improved as banks tend to keep their liquidity levels at their original targets, which are now their optimal ones, but of course, at the cost of the central bank.

Now let's consider the effect of the penalty parameter $\bar{q}$. The results show that when the penalty of deviating from the target is high, banks implement more aggressive control policy. Specifically, as the penalty parameter $\bar{q}$ increases, banks hold more liquidity provision for the interbank relationship, and increase the adjustment speeds toward their optimal targets to keep their liquidity levels closer to the targets. The relationship between the precautionary liquidity and the penalty parameter is unclear. However, as the penalty parameter $\bar{q}$ approaches infinity, banks set the optimal targets to their original targets (zero precautionary liquidity) but with adjustment speeds approaching infinity, and hold liquidity provision at half of their total interbank exposures, or $\frac{\bar{\zeta}}{2}$. In other words, if the penalty cost is large enough, banks aim to keep their liquidity levels at their original targets using the highest adjustment speeds, and at the same time try to reduce the impact from the interbank relationship by half. This type of banks' control policies should make the banking system more stable, but, unlike the case of using the reward parameter $\bar{b}$, it comes at the cost of the banks. So the question is would this excessive liquidity control cost lead to other types of systemic risk in the banking system?

5 Core-periphery network with group homogeneity and multiple regimes

In this section we consider a multiple-regime core-periphery interbank network with a finite number of banks. This network consists of core and peripheral banks. Core banks represent important banks in the interbank network and are connected to each other. Peripheral banks, on the other hand, are smaller banks and are connected to core banks, but not any other peripheral banks. We assume that banks of the same type are homogeneous. This *group homogeneity* assumption simplifies the structure of the value functions, and hence allows us to study the structure of the optimal controls. The interbank network is subject to changes between regimes. We first provide the forms of the value function and the optimal control policies, and use numerical examples with two regimes to study how the optimal liquidity controls respond to various scenarios including (i) the changes in the interbank interconnectedness during crises due to changes in the network topology and the overall network exposure, (ii) systemic shocks representing, for example, bank runs, and (iii) panic-triggered flows which could represent a flight-to-quality effect. We also study the effects of the incentive and penalty parameters on the optimal control policy. Finally, we use simulation to study the systemic risk of the interbank network and draw some policy implications.

5.1 Value function and optimal control

In the core-periphery network, banks are divided into two groups: core and periphery. We assume that banks are homogeneous within each group. Specifically, let C denote the set of core banks, and P the set of peripheral banks. We assume there are M_C core banks and M_P peripheral banks, and that $\theta_i = \bar{\theta}_G, \sigma_i = \bar{\sigma}_G, \eta_i = \bar{\eta}_G, \mu_i = \bar{\mu}_G, b_i = \bar{b}_G, q_i = \bar{q}_G$, and $\delta_i = \bar{\delta}_G$ for all banks i in group G where $G \in \{C, P\}$. Moreover, the lending preference π_{ij} depends only on the group of bank i and that of bank j. Let $\bar{\pi}_C, \bar{\pi}_P$ and $\bar{\pi}_{CP}$ denote the lending preferences between core banks, between peripheral banks, and between a core bank and a peripheral bank, respectively. These parameters and the network exposure ζ are regime-dependent. Under the group homogeneity assumption the dynamic (8) of a core bank i can be rewritten as

$$\begin{aligned} dX_i(t) &= \alpha_i(t)dt + \zeta(Y(t))(M_C-1)\bar{\pi}_C(Y(t))\Big\{\bar{X}_{C_{-i}}(t) - X_i(t)\Big\}dt \\ &\quad + \zeta(Y(t))M_P\bar{\pi}_{CP}(Y(t))\Big\{[\bar{X}_P(t) - \bar{\theta}_P(Y(t))] - [X_i(t) - \bar{\theta}_C(Y(t))]\Big\}dt \\ &\quad + \bar{\sigma}_C(Y(t))\sqrt{X_i(t)}dW_i(t) + \sum_{z=1}^{K}\bar{\eta}_C(Y(t^-),z)X_i(t^-)dN_z(t) \\ &\quad + \bar{\mu}_C(Y(t))[X_i(t) - \bar{\theta}_C(Y(t))]dt, \end{aligned} \tag{71}$$

and that of a peripheral bank i can be rewritten as

$$\begin{aligned} dX_i(t) &= \alpha_i(t)dt + \zeta(Y(t))(M_P-1)\bar{\pi}_P(Y(t))\Big\{\bar{X}_{P_{-i}}(t) - X_i(t)\Big\}dt \\ &\quad + \zeta(Y(t))M_C\bar{\pi}_{CP}(Y(t))\Big\{[\bar{X}_C(t) - \bar{\theta}_C(Y(t))] - [X_i(t) - \bar{\theta}_P(Y(t))]\Big\}dt \\ &\quad + \bar{\sigma}_P(Y(t))\sqrt{X_i(t)}dW_i(t) + \sum_{z=1}^{K}\bar{\eta}_P(Y(t^-),z)X_i(t^-)dN_z(t) \\ &\quad + \bar{\mu}_P(Y(t))[X_i(t) - \bar{\theta}_P(Y(t))]dt \end{aligned} \tag{72}$$

where C_{-i} denote the set of all core banks except bank i, P_{-i} the set of all peripheral banks except bank i, and $\bar{X}_G(t)$ is the average of the liquidity levels of the banks in set G. As we can see, the group homogeneity assumption simplifies the effect of the interbank network on the liquidity dynamic. More specifically, the interbank network effect can be summarized by the current liquidity of the bank and the two averages: the average of the peer banks within the same group, and the average of the banks in the other group. Under this model, we obtain the following HJB equations: for each $i \in C$ and $y = 1, \ldots, K$

$$\begin{aligned} \min_{\alpha_i}\Bigg\{ &\sum_{j\in C}\frac{\partial V_i(x,y)}{\partial x_j}\Big(\alpha_i \mathbf{1}_{j=i} + \hat{\alpha}_j(x,y)\mathbf{1}_{j\neq i} + \bar{\mu}_C(y)[x_j - \bar{\theta}_C(y)] \\ &\qquad + \zeta(y)\Big[(M_C-1)\bar{\pi}_C(y)(\bar{x}_{C_{-j}} - x_j) + M_P\bar{\pi}_{CP}(y)\big([\bar{x}_P - \bar{\theta}_P(y)] - [x_j - \bar{\theta}_C(y)]\big)\Big]\Big) \\ &+ \sum_{j\in P}\frac{\partial V_i(x,y)}{\partial x_j}\Big(\hat{\alpha}_j(x,y) + \bar{\mu}_P(y)[x_j - \bar{\theta}_P(y)] \\ &\qquad + \zeta(y)\Big[(M_P-1)\bar{\pi}_P(y)(\bar{x}_{P_{-j}} - x_j) + M_C\bar{\pi}_{CP}(y)\big([\bar{x}_C - \bar{\theta}_C(y)] - [x_j - \bar{\theta}_P(y)]\big)\Big]\Big) \\ &+ \frac{1}{2}\sum_{j\in C}\frac{\partial^2 V_i(x,y)}{\partial x_j^2}\bar{\sigma}_C^2(y)x_j + \frac{1}{2}\sum_{j\in P}\frac{\partial^2 V_i(x,y)}{\partial x_j^2}\bar{\sigma}_P^2(y)x_j + \sum_{z=1}^{K}\lambda(y,z)\left[V_i(x + \bar{\eta}(y,z)x, z) - V_i(x,y)\right] \\ &+ \frac{1}{2}\alpha_i^2 - \bar{b}_C(y)\alpha_i(\bar{\theta}_C(y) - x_i) + \frac{1}{2}\bar{q}_C(y)(x_i - \bar{\theta}_C(y))^2 - \bar{\delta}_C(y)V_i(x,y)\Bigg\} = 0 \end{aligned} \tag{73}$$

and for each $i \in P$ and $y = 1, \dots, K$

$$\min_{\alpha_i}\Bigg\{ \sum_{j\in C} \frac{\partial V_i(x,y)}{\partial x_j}\Big(\hat{\alpha}_j(x,y) + \bar{\mu}_C(y)[x_j - \bar{\theta}_C(y)] + \zeta(y)\Big[(M_C-1)\bar{\pi}_C(y)(\bar{x}_{C_{-j}} - x_j) + M_P\bar{\pi}_{CP}(y)\big([\bar{x}_P - \bar{\theta}_P(y)] - [x_j - \bar{\theta}_C(y)]\big)\Big]\Big)$$
$$+ \sum_{j\in P} \frac{\partial V_i(x,y)}{\partial x_j}\Big(\alpha_i \mathbf{1}_{j=i} + \hat{\alpha}_j(x,y)\mathbf{1}_{j\neq i} + \bar{\mu}_P(y)[x_j - \bar{\theta}_P(y)] + \zeta(y)\Big[(M_P-1)\bar{\pi}_P(y)(\bar{x}_{P_{-j}} - x_j) + M_C\bar{\pi}_{CP}(y)\big([\bar{x}_C - \bar{\theta}_C(y)] - [x_j - \bar{\theta}_P(y)]\big)\Big]\Big)$$
$$+ \frac{1}{2}\sum_{j\in C}\frac{\partial^2 V_i(x,y)}{\partial x_j^2}\bar{\sigma}_C^2(y)x_j + \frac{1}{2}\sum_{j\in P}\frac{\partial^2 V_i(x,y)}{\partial x_j^2}\bar{\sigma}_P^2(y)x_j + \sum_{z=1}^{K}\lambda(y,z)\left[V_i(x+\bar{\eta}(y,z)x,z) - V_i(x,y)\right]$$
$$+ \frac{1}{2}\alpha_i^2 - \bar{b}_P(y)\alpha_i(\bar{\theta}_P(y) - x_i) + \frac{1}{2}\bar{q}_P(y)(x_i - \bar{\theta}_P(y))^2 - \bar{\delta}_P(y)V_i(x,y)\Bigg\} = 0. \quad (74)$$

It can be shown that the value function of a core bank i when $X_i(t) = x_i$, $\bar{X}_{C_{-i}}(t) = \bar{x}_{C_{-i}}$, $\bar{X}_P(t) = \bar{x}_P$ and $Y(t) = y$ is given by

$$\begin{aligned} V(x_i, \bar{x}_{C_{-i}}, \bar{x}_P, y) = {} & \frac{1}{2}\gamma_1^C(y)(x_i - \hat{\theta}_C(y))^2 + \frac{1}{2}\gamma_2^C(y)(\bar{x}_{C_{-i}} - x_i)^2 \\ & + \frac{1}{2}\gamma_2^{CP}(y)([\bar{x}_P - \bar{\theta}_P(y)] - [x_i - \bar{\theta}_C(y)])^2 \\ & + \frac{1}{2}\gamma_3^C(y)(\bar{x}_{C_{-i}} - \tilde{\theta}_C(y))^2 + \frac{1}{2}\gamma_3^{CP}(y)(\bar{x}_P - \tilde{\theta}_{CP}(y))^2 \\ & + \frac{1}{2}\gamma_4^C(y)([\bar{x}_{C_{-i}} - \bar{\theta}_C(y)] - [\bar{x}_P - \bar{\theta}_P(y)])^2 + \gamma_5^C(y) \end{aligned} \quad (75)$$

where $\gamma_1^C, \gamma_2^C, \gamma_2^{CP}, \gamma_3^C, \gamma_3^{CP}, \gamma_4^C, \gamma_5^C, \hat{\theta}_C, \tilde{\theta}_C$ and $\tilde{\theta}_{CP}$ are regime-dependent parameters which can be obtained from solving a system of equations similar to that of the general case (18) - (20). Observe that the value function of a core bank i depends on six types of distance. The first two are the distances between the current liquidity level of bank i and (i) an *adjusted* liquidity target $\hat{\theta}_C$, and (ii) the liquidity average of the other core banks $\bar{X}_{C_{-i}}$. The next one is the difference between the excess liquidity of the bank $X_i - \bar{\theta}_C$ and the average excess liquidity of the peripheral banks $\bar{X}_P - \bar{\theta}_P$. The next two are the distances between each of the averages $\bar{X}_{C_{-i}}$ and $\bar{X}_P$ and its reference level $\tilde{\theta}_C$ and $\tilde{\theta}_{CP}$, respectively. The last one is the distance between the two averages of excess liquidity $\bar{X}_{C_{-i}} - \bar{\theta}_C$ and $\bar{X}_P - \bar{\theta}_P$.

Similarly, the value function of a peripheral bank i when $X_i(t) = x_i$, $\bar{X}_{P_{-i}}(t) = \bar{x}_{P_{-i}}$, $\bar{X}_C(t) =$

$\bar{x}_C$ and $Y(t) = y$ is given by

$$
\begin{aligned}
V(x_i, \bar{x}_{P_{-i}}, \bar{x}_C, y) = &= \frac{1}{2}\gamma_1^P(y)(x_i - \hat{\theta}_P(y))^2 + \frac{1}{2}\gamma_2^P(y)(\bar{x}_{P_{-i}} - x_i)^2 \\
&+ \frac{1}{2}\gamma_2^{PC}(y)([\bar{x}_C - \bar{\theta}_C(y)] - [x_i - \bar{\theta}_P(y)])^2 \\
&+ \frac{1}{2}\gamma_3^P(y)(\bar{x}_{P_{-i}} - \tilde{\theta}_P(y))^2 + \frac{1}{2}\gamma_3^{PC}(y)(\bar{x}_C - \tilde{\theta}_{PC}(y))^2 \\
&+ \frac{1}{2}\gamma_4^P(y)([\bar{x}_{P_{-i}} - \bar{\theta}_P(y)] - [\bar{x}_C - \bar{\theta}_C(y)])^2 + \gamma_5^P(y). \qquad (76)
\end{aligned}
$$

From (16), we have the optimal control for a core bank i when $X_i(t) = x_i$, $\bar{X}_{C_{-i}}(t) = \bar{x}_{C_{-i}}$, $\bar{X}_P(t) = \bar{x}_P$ and $Y(t) = y$ is

$$
\begin{aligned}
\hat{\alpha}_i(x_i, \bar{x}_{C_{-i}}, \bar{x}_P, y) &= b_C(y)(\bar{\theta}_C(y) - x_i) + \gamma_1^C(y)(\hat{\theta}_C(y) - x_i) \\
&+ \gamma_2^C(y)(\bar{x}_{C_{-i}}(y) - x_i) + \gamma_2^{CP}(y)([\bar{x}_P(y) - \bar{\theta}_P(y)] - [x_i - \bar{\theta}_C(y)]) \qquad (77)
\end{aligned}
$$

and the optimal control for a peripheral bank i when $X_i(t) = x_i$, $\bar{X}_{P_{-i}}(t) = \bar{x}_{P_{-i}}$, $\bar{X}_C(t) = \bar{x}_C$ and $Y(t) = y$ is

$$
\begin{aligned}
\hat{\alpha}_i(x_i, \bar{x}_{P_{-i}}, \bar{x}_C, y) &= b_P(y)(\bar{\theta}_P(y) - x_i) + \gamma_1^P(y)(\hat{\theta}_P(y) - x_i) \\
&+ \gamma_2^P(y)(\bar{x}_{P_{-i}}(y) - x_i) + \gamma_2^{PC}(y)([\bar{x}_C(y) - \bar{\theta}_C(y)] - [x_i - \bar{\theta}_P(y)]). \qquad (78)
\end{aligned}
$$

This implies that banks control their liquidity levels based on their liquidity levels relative to their original targets and adjusted targets, and the differences between the deviations of their liquidity levels from the original targets and those of the peer averages in the same group, and in the other group. As the targets between banks can be different in this core-periphery network structure, we explicitly call γ_2 as the *proportional liquidity provision* for the interbank relationship.

As mentioned earlier, explicit solutions for this multiple-regime core-peripheral network model do not exist. In the subsequent sections, we provide numerical examples, and discuss the results.

5.2 Benchmark model parameters

To study the impacts of different sources of risk on optimal control policies and systemic risk, we consider a core-periphery network with two regimes, which are normal regime ($Y = 1$) and crisis regime ($Y = 2$), and vary different parameter values including the network topology. In this section we discuss how our baseline parameter values are chosen for our benchmark models.

In our study, we choose the model parameters to represent a banking system in which there are a small number of core banks and a relatively larger number of peripheral banks, but the interbank network exposures of the core banks significantly dominate those of the peripheral banks. Core banks represent large banks and peripheral banks represent small banks. This network characteristic is found in many countries. For example, Veld and Lelyveld (2014) find the core-periphery structure in the interbank markets of the Netherlands. In our network, we assume that there are 4 core banks and 24 peripheral banks, so that a ratio between core and peripheral banks is 1 : 6. We assume that the liquidity targets $\bar{\theta}_C$ and $\bar{\theta}_P$ are proportional to the total assets of the banks. Assuming that the total asset from all of the large banks accounts for around 85% of the banking sector, we choose $\bar{\theta}_C$ and $\bar{\theta}_P$ to be 72 and 2, respectively. We assume further that these liquidity targets are twice as large as the required liquidity reserve, and that the regulator sets the required liquidity reserve level, which is $\theta/2$, to be three times of the standard deviation of the idiosyncratic shocks from the daily deposits and withdrawals when the liquidity is at the required reserve level during the normal times. That is, $\frac{\bar{\theta}(1)}{2} = 3\sigma(1)\sqrt{\frac{\bar{\theta}(1)}{2}}$. Consequently, we set $\bar{\sigma}_C(1) = \frac{1}{3}\sqrt{\frac{72}{2}} = 2$ and $\bar{\sigma}_P(1) = \frac{1}{3}\sqrt{\frac{2}{2}} = \frac{1}{3}$. The volatility parameters $\bar{\sigma}$ are assumed to be doubled during crises: $\bar{\sigma}_C(2) = 4$ and $\bar{\sigma}_P(2) = \frac{2}{3}$.

We set the lending preference between core banks or $\bar{\pi}_C$ to 1. The value of the lending preference between core and peripheral banks or $\bar{\pi}_{CP}$ is set to 1/64 so that the average interbank exposure of a core bank is around 100 times larger than that of a peripheral bank where the average exposure of a core bank is $\zeta(M_C - 1)\bar{\pi}_C\bar{\theta}_C + \zeta M_P\bar{\pi}_{CP}(\bar{\theta}_C + \bar{\theta}_P)/2$, and the average exposure of a peripheral bank is $\zeta M_C\bar{\pi}_{CP}(\bar{\theta}_C + \bar{\theta}_P)/2$. This number is in the neighborhood of the ratio between the interbank lending of a core and a peripheral bank of Craig and Ma (2021) (German data), and Lin and Zhang (2021) (Chinese data). As the peripheral banks are not connected directly to each other, we set $\bar{\pi}_P = 0$. The overall network exposure or ζ is set to 2.1 so that when the liquidity level of a core bank is not at its target, it can use the interbank network with the other core banks to move it towards the target by reducing the distance between the current liquidity level and the target by 80% within one quarter assuming that the liquidity levels of all of the other core banks are always at their liquidity targets. That is, we choose ζ so that

$$e^{-\zeta(M_C-1)\bar{\pi}_C\times\frac{1}{4}} = \frac{\bar{\theta}_C - X_i(t+\frac{1}{4})}{\bar{\theta}_C - X_i(t)} \approx 0.2. \tag{79}$$

The lending preferences and the overall network exposure under the baseline model are assumed to be the same in both regimes.

The sizes of systemic liquidity shocks or $\bar{\eta}$ that happen when the regime transitions from the normal regime to the crisis regime are set to −10% for core banks and −30% for peripheral banks. This could represent a systemic bank run for which the effects on the smaller and hence viewed as riskier banks are larger. During the crisis regime, it is assumed that within-regime shocks lead to −1% change of the liquidity of each core bank, and it is −3% for each peripheral bank, or 10 times smaller than their respective regime-transition shocks. We assume that there are no systemic shocks within the normal regime ($\bar{\eta}(1,1) = 0$) nor the regime-transition shocks when the regime switches from the crisis regime to the normal regime ($\bar{\eta}(2,1) = 0$). We set the transition rate from the normal regime to the crisis regime at $\lambda(1,2) = 0.5$ which corresponds to an average time in a normal regime of two years, while we set $\lambda(2,1) = 2$ which corresponds to an average time in a crisis regime of half a year. The arrival rate of systemic shocks within the normal regime is set to 0, and that within the crisis regime is set to 8 times a year.

It is important to note that the values of the targets $\bar{\theta}$ and the sizes of crisis-triggering shock $\bar{\eta}(1,2)$ and within-crisis shock $\bar{\eta}(2,2)$ imply a larger reduction in the liquidity level of the core banks than the peripheral banks when their liquidity levels are the their original targets at the time when the shock occurs ($|\bar{\eta}_C(1,2)|\bar{\theta}_C > |\bar{\eta}_P(1,2)|\bar{\theta}_P$ and $|\bar{\eta}_C(2,2)|\bar{\theta}_C > |\bar{\eta}_P(2,2)|\bar{\theta}_P$). This leads to positive liquidity demands from the core banks via the interbank markets after the shocks. This could represent the scenario in which core banks have net positive interbank lending balances during the normal regime while the peripheral banks hold net negative intebank lending balances, but once the crisis occurs, the core banks stop lending and require their net borrowers to return the borrowed liquidity. The effect continues further as within-regime shocks arrive. As a result, we could expect to see liquidity to flow from the peripheral banks to the core banks via the interbank market at the start of the crisis regime, as well as during the crisis regime.

The flight-to-quality effect can represent the flows of liquidity due to depositors withdrawing their money from low-quality banks or depositing their money to high-quality banks. We set the associated parameter $\bar{\mu}$ to 0 for the normal regime. For the crisis regime, the number is chosen so that the effect of the flight-to-quality flows is equal to the effect from the within-regime shock but with 25% smaller shock size. More specifically, consider a peripheral bank. Assume that there is a regime change from the normal regime to the crisis regime at time t with $X_i(t^-) = \bar{\theta}_P$. Then the liquidity level of bank i right after the regime transition is $X_i(t) = (1+\bar{\eta}(1,2))\bar{\theta}_P$. The effect from the shocks within the crisis regime over the next one month, assuming that the regime does not change over that time period and ignoring all of the other factors in the dynamic of $X_i(t)$, would bring the liquidity level to $X_i(t+\frac{1}{12}) = (1+\bar{\eta}(2,2))^{N_2(t+\frac{1}{12})-N_2(t)} \times (1+\bar{\eta}(1,2))\bar{\theta}_P$ where $N_2(t+\frac{1}{12}) - N_2(t)$ is the number of within-regime shocks over the one-month period. With 25% smaller shock size, we choose $\bar{\mu}$ so that

$$e^{\bar{\mu}_P(2)\times\frac{1}{12}} \approx \frac{\bar{\theta}_P - \mathbb{E}\left[(1+0.75\bar{\eta}_P(2,2))^{N_2(t+\frac{1}{12})-N_2(t)} \times (1+\bar{\eta}_P(1,2))\bar{\theta}_P\right]}{\bar{\theta}_P - (1+\bar{\eta}_P(1,2))\bar{\theta}_P} \tag{80}$$

$$= \frac{1 - e^{-\frac{1}{12}\lambda(2,2)(-0.75\bar{\eta}_P(2,2))}(1+\bar{\eta}_P(1,2))}{-\bar{\eta}_P(1,2)}. \tag{81}$$

We set the value of $\bar{\mu}$ for the core banks equal to that for the peripheral banks in the crisis regime.

The value of $\bar{q}$ determines the cost associated with a deviation of the liquidity from the target. We choose the value of $\bar{q}$ so that the total cost of taking no action and leaving the liquidity level equal to $k\bar{\theta}$ for some $k > 0, k \neq 1$, for one year is twice as much as the total cost of moving the liquidity level from $k\bar{\theta}$ back to $\bar{\theta}$ within one year using a constant α, ignoring the reward from the incentive parameter $\bar{b}$ and the discount rate $\bar{\delta}$. More precisely, the cost of leaving the liquidity level at $k\bar{\theta}$ for one year is

$$Cost_{\text{no action}} = \int_0^1 \frac{\bar{q}}{2}(\bar{\theta} - k\bar{\theta})^2 dt = \frac{\bar{q}}{2}(1-k)^2\bar{\theta}^2. \tag{82}$$

The constant α that takes the liquidity from $k\bar{\theta}$ back to $\bar{\theta}$ within one year is equal to $\alpha = (1-k)\bar{\theta}$. As a result, the value of $X_i(t)$ for $t \in [0,1]$ is $X_i(t) = X_i(0) + \alpha t = k\bar{\theta} + (1-k)\bar{\theta}t$. So the total cost of constantly moving the liquidity back to the target is

$$Cost_{\text{constant } \alpha} = \int_0^1 \left[\frac{1}{2}((1-k)\bar{\theta})^2 + \frac{\bar{q}}{2}(\bar{\theta} - (k\bar{\theta} + (1-k)\bar{\theta}t))^2\right] dt = \frac{1}{2}(1-k)^2\bar{\theta}^2 + \frac{\bar{q}}{6}(1-k)^2\bar{\theta}^2. \tag{83}$$

Assuming $Cost_{\text{no action}} = 2 \times Cost_{\text{constant } \alpha}$, we obtain $\bar{q} = 6$. We use this number for all banks and all regimes.

As the incentive parameter $\bar{b}$ can be viewed as the saving in the lending rates that the central bank provides to commercial banks through its lending facilities, we choose the value of $\bar{b}$ to reflect the risk of each bank type. We choose $\bar{b}_C = 0.7$ and $\bar{b}_P = 0.2$ which is equivalent to providing the incentive to core banks to reduce their liquidity gap by around 50% over one year, and reduce it by around 20% for peripheral banks: $e^{-\bar{b}_C \times 1} \approx 0.5$ and $e^{-\bar{b}_P \times 1} \approx 0.8$. Finally, we choose the discount rate $\bar{\delta} = 1\%$ for all banks and regimes. Table 1 summarizes all of the parameter values.

The above benchmark model assumes that the core-periphery structure of the interbank network is preserved in the crisis regime. We refer to this as *core-periphery* benchmark model. Since the interbank relationships and activities may slowdown or collapse when a systemic liquidity crisis occurs, we introduce a *no-network* benchmark model to study the effects of the collapse of the interbank network during crises by letting $\pi_{ij}(2) = 0$ for all i, j.

5.3 Optimal control under benchmark models

Table 2 reports the optimal control parameters under the core-periphery (Panel A) and no-network (Panel B) benchmark models. Observe first that γ_1 and γ_2 from both models have the same sign as those in the large homogeneous complete network with one regime discussed in Section 4 ($\gamma_1 \geq 0, \gamma_2 \leq 0$). We also report the *proportional* precautionary liquidity $\frac{\Delta\bar{\theta}}{\bar{\theta}}$ to account for differences in the original targets of core and peripheral banks instead of $\Delta\bar{\theta}$. Under the core-periphery model (Panel A), core banks adjust their liquidity levels toward their optimal targets slower than peripheral banks for both regimes mainly because the total interconnectedness level ($\zeta \sum_j \pi_{i,j}$) of a core bank is much larger than that of a peripheral bank (7.088 for a core bank, and 0.131 for a peripheral bank), and as a consequence, the benefit of adjusting their liquidity levels toward their targets can be less if it goes against a peer average in a strong interbank relationship. This is consistent with the result of the one-regime model in Section 4. Comparing the adjustment speed γ_1 between the normal and crisis regimes, we find that both

Parameter	Core banks		Peripheral banks	
	Normal regime ($y=1$)	Crisis regime ($y=2$)	Normal regime ($y=1$)	Crisis regime ($y=2$)
General characteristics				
M	4	4	24	24
$\bar{\theta}(y)$	72	72	2	2
$\bar{\sigma}(y)$	2	4	1/3	2/3
Interbank network				
$\bar{\pi}_C(y)$	1	1		
$\bar{\pi}_{CP}(y)$	1/64	1/64	1/64	1/64
$\bar{\pi}_P(y)$			0	0
$\zeta(y)$	2.1	2.1	2.1	2.1
Systemic shocks				
$\bar{\eta}(1,y)$	0	-0.10	0	-0.30
$\bar{\eta}(2,y)$	0	-0.01	0	-0.03
$\lambda(1,y)$	0	0.5	1	0.5
$\lambda(2,y)$	2	8	2	8
Flight-to-quality				
$\bar{\mu}(y)$	0	0.4	0	0.4
Objective function				
$\bar{q}(y)$	6	6	6	6
$\bar{b}(y)$	0.7	0.7	0.2	0.2
$\bar{\delta}(y)$	0.01	0.01	0.01	0.01

Table 1

Parameter values of the core-periphery benchmark model core and peripheral banks increase their adjustment speeds upward during the crisis regime. This is possibly to account for the effect of flight-to-quality flows in the crisis regime.

	Core banks ($G = C, G' = P$)		Peripheral banks ($G = P, G' = C$)	
	Normal regime ($y = 1$)	Crisis regime ($y = 2$)	Normal regime ($y = 1$)	Crisis regime ($y = 2$)
Panel A: Core-periphery benchmark				
$\gamma_1^G(y)$	0.761	0.825	2.095	2.277
$b_G(y) + \gamma_1^G(y)$	1.461	1.525	2.295	2.477
$\gamma_2^G(y)$	−0.274	−0.306	−0.008	−0.009
$\gamma_2^{GG'}(y)$	−0.075	−0.084	−0.057	−0.064
$\Delta\bar{\theta}_G(y)/\bar{\theta}_G(y)$	0.004	0.004	0.045	0.059
Panel B: No-network benchmark				
$\gamma_1^G(y)$	0.789	1.613	2.095	2.311
$b_G(y) + \gamma_1^G(y)$	1.489	2.313	2.295	2.511
$\gamma_2^G(y)$	−0.277	−0.094	−0.007	−0.002
$\gamma_2^{GG'}(y)$	−0.074	−0.023	−0.054	−0.017
$\Delta\bar{\theta}_G(y)/\bar{\theta}_G(y)$	0.006	−0.002	0.034	0.045

Table 2

Optimal control policy under the core-periphery and no-network benchmark models

For the proportional liquidity provision for interbank relationship (γ_2), core banks hold more provision (proportional to the gaps of excess liquidity) for other core banks than for peripheral banks ($\gamma_2^C > \gamma_2^{CP}$), while it is opposite for peripheral banks ($\gamma_2^P < \gamma_2^{PC}$) for both regimes. This is simply because the total level of interconnectedness between core banks is higher than that between core and peripheral banks, and there is no direct connectivity between peripheral banks. However, peripheral banks still maintain small liquidity provision for other peripheral banks ($\gamma_2^P < 0$) due to an indirect effect from other peripheral banks who lend liquidity to or borrow liquidity from core banks and pass it through the other peripheral banks. Comparing between the two regimes, we find that banks hold more liquidity provision during the crisis regime. Again, this could come from the effect of flight-to-quality flows in the crisis regime.

For the proportional precautionary liquidity ($\Delta\bar{\theta}/\bar{\theta}$), we find that both types of banks hold positive precautionary liquidity in both regimes, implying that the effect from the liquidity shock (positive precautionary) seems to outweigh the effect from the volatility (negative precautionary). As we can see, the proportional precautionary of the core banks is less than that of peripheral banks because core banks have smaller relative size of liquidity shock ($\bar{\eta}$) than peripheral banks, but both core and peripheral banks have the same ratio of the variance to the target ($\bar{\sigma}^2/\bar{\theta}$).

In the crisis regime, the effect of the liquidity shocks increases due to the higher arrival rate of shocks, but at the same time the volatility is doubled. There is also an additional effect from the flight-to-quality flows in the crisis regime. So the overall effect on the precautionary liquidity is unclear. As it turns out, core banks hold approximately the same level of precautionary liquidity in the normal and crisis regimes (in fact a little smaller level in the crisis regime), while peripheral banks hold more precautionary liquidity in the crisis regime than in the normal regime.

Panel B of Table 2 reports the optimal control parameters for the no-network benchmark model. When the interbank network collapses during the crisis regime, banks adjust their liquidity toward their optimal targets at approximately the same speed in the normal regime, but the speed is much faster in the crisis regime, particularly for core banks, compared to the case in which the network structure is preserved. More specifically, the total adjustment speed $\bar{b}+\gamma_1$ for core banks significantly increases from 1.525 in the core-periphery structure to 2.313 when the interbank network collapses, while it increases from 2.477 to 2.511 for the peripheral banks. This result could be explained by noting that the benefit of controlling their liquidity is more when there is a reduction in the interconnectedness level in the crisis regime, and because the total level of interconnectedness of the core banks is much larger than that of the peripheral banks when the network structure is preserved, when the network collapses, the impact on the core banks is larger.

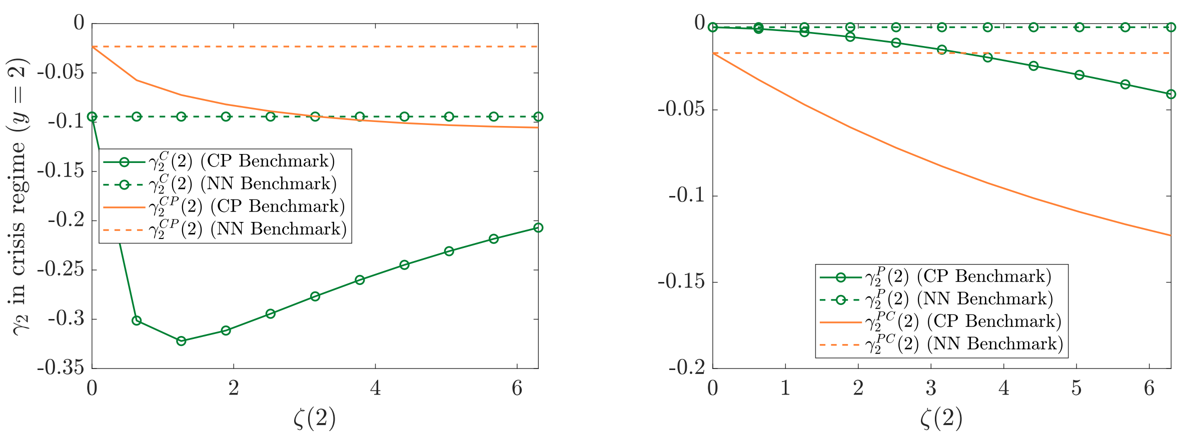
The concerns about liquidity provision for interbank relationship in the crisis regime are also reduced as we can see from the reduction in the magnitude of γ_2. As mentioned earlier, the liquidity needs are expected to be high given there is the flight-to-quality effect in the crisis regime, but the interbank market cannot function in the crisis which means that there is no needs for supplying liquidity via the interbank market. Hence, banks tend to lower their proportional liquidity provision (γ_2).

As discussed earlier, the values of the targets $\bar{\theta}$ and the crisis-triggering shock $\bar{\eta}(1,2)$ imply a likely larger reduction in the liquidity level of the core banks than the peripheral banks ($|\bar{\eta}_C(1,2)|\bar{\theta}_C > |\bar{\eta}_P(1,2)|\bar{\theta}_P$). This causes the liquidity in the interbank market to flow from the peripheral banks to the core banks. When the interbank market collapses during the crisis regime, this channel of liquidity supply of the core banks disappears, making the core banks to hold a slightly more precautionary liquidity during the normal regime (from 0.004 to 0.006). The effect on the peripheral banks is opposite, so they hold less precautionary liquidity in the normal regime (reduces from 0.045 to 0.034). Finally, the proportional precautionary liquidity of both core and peripheral banks in the crisis regime reduce when the interbank market collapses than when it still functions, as the volatility effect becomes more important (γ_2 is closer to zero).

In this section, we observe general characteristics of optimal liquidity control policies under the benchmark models, and the effect of the change in the network topology during the crisis for the baseline model parameters. The analysis does not yet provide some other features of the control policies with respect to changing risk parameters. In the next section we discuss the sensitivities of the control policies to the changes in risk parameters associated with the models, including the interconnectedness level ζ, the size of systemic liquidity shocks $\bar{\eta}$ and the effect of the flight-to-quality flows $\bar{\mu}$ in the crisis regime. The effects of the reward ($\bar{b}$) and penalty parameters ($\bar{q}$) on the optimal control policies are also discussed.

5.4 Effects of interconnectedness level

In this section we vary the value of the overall interbank network exposure in the crisis regime or $\zeta(2)$ from 0 to 3 times of the value in the baseline. A high level of interconnectedness could reflect a low-friction interbank market and/or high interbank activities and commitments, while a low level could represent liquidity hoarding during the crisis regime.



(a) Core bank (b) Peripheral bank

Figure 1. Values of proprotional liquidity provision for the interbank relationship in the crisis regime of core banks ($\gamma_2^C, \gamma_2^{CP}$) (left) and of peripheral banks ($\gamma_2^P, \gamma_2^{PC}$) (right) associated with the core-periphery (CP) benchmark (solid line) and the no-network (NN) benchmark (dashed line) models as a function of the overall network exposure in the crisis regime ($\zeta(2)$).

Figure 1 shows the proportional liquidity provision in the crisis regime for the interbank relationship (γ_2) for core banks (left) and peripheral banks (right) for both core-periphery (CP) benchmark model (solid lines) and no-network (NN) benchmark model (dashed lines). In the NN model, the lending preferences in the crisis regime $\pi_{ij}(2) = 0$ for all i, j and therefore the liquidity provision does not vary as $\zeta(2)$ changes. From the left panel, we can see that core banks hold more proportional liquidity provision for the other core banks (lines with circles) in absolute terms than that for the peripheral banks (lines without circles) due to the higher total interconnectedness between core banks as observed from earlier analysis. The value of $\gamma_2^C(2)$ keeps decreasing as $\zeta(2)$ getting larger at low levels of $\zeta(2)$, and it becomes increasing when the level of $\zeta(2)$ is sufficiently large. This response pattern of the core banks against their peers within the core group is similar to the response pattern in the single-regime homogeneous banks environment in Section 4: when the interconnectedness level is large enough, the benefits of reducing its effect using the liquidity provision is less. The right panel of Figure 1 shows the proportional liquidity provision for peripheral banks. Interestingly, the peripheral banks hold more proportional provision as $\zeta(2)$ increases for both of the provisions for the core banks and for the other peripheral banks even though there is no direct linkage between the peripheral banks. As $\zeta(2)$ increases, the interbank channels between core and peripheral banks provide larger indirect effects between the peripheral banks.

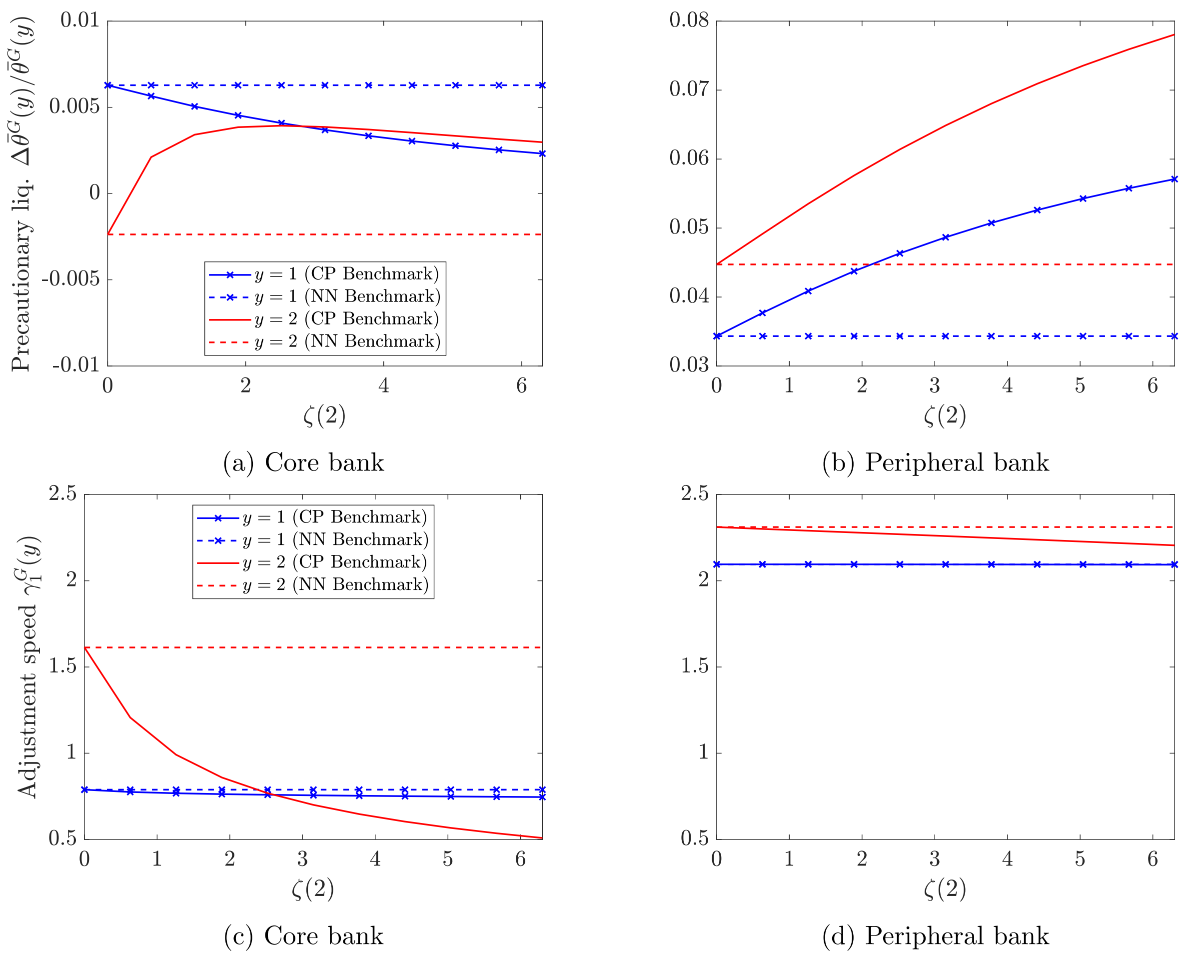


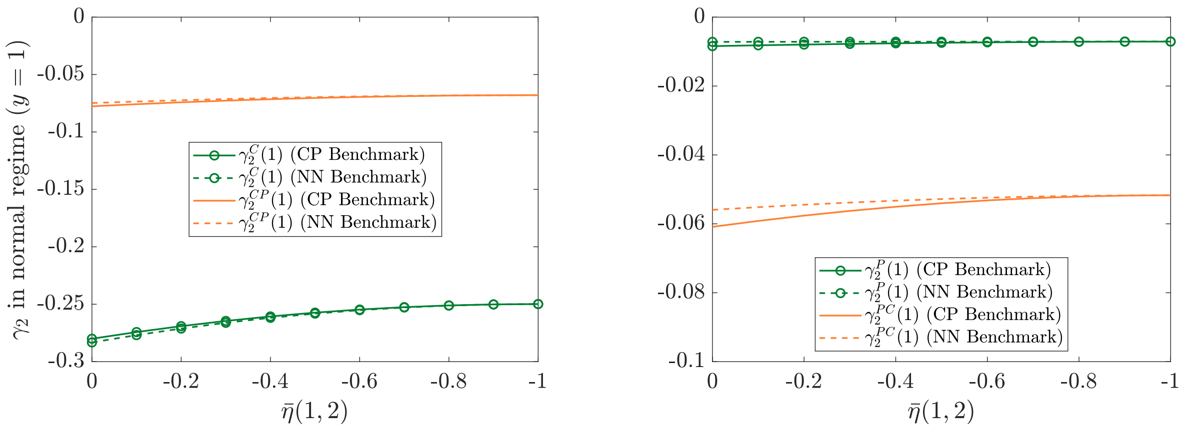
Figure 2. Values of proportional precautionary liquidity ($\Delta\bar{\theta}/\bar{\theta}$) (upper) and values of adjustment speed γ_1 (lower) for core (left) and peripheral (right) banks associated with the core-periphery (CP) benchmark (solid line) and the no-network (NN) benchmark (dashed line) models as a function of the overall network exposure in the crisis regime ($\zeta(2)$). The values are for normal (lines with crosses) and crisis (lines without crosses) regimes.

Figure 2 shows the effects of $\zeta(2)$ on the proportional precautionary liquidity $\Delta\bar{\theta}/\bar{\theta}$ (upper) and the non-incentive component of the adjustment speed γ_1 (lower) for core banks (left) and peripheral banks (right). As we can see from the core-periphery benchmark model (solid lines), the proportional precautionary liquidity of core banks in the normal regime is positive (the shock's effect is larger than the volatility's effect) and declines when $\zeta(2)$ increases, while their adjustment speed slightly decreases. The decrease in both policy parameters show that as the interconnectedness level during the crisis regime is higher, the benefit for liquidity reservation in the normal regime is less since their excess liquidity would eventually be shared to the peers' average more quickly, especially among the core group itself. However, for the crisis regime, when $\zeta(2)$ is small, core banks hold negative precautionary liquidity as the volatility's effect is larger than the shock's effect, and as $\zeta(2)$ increases, the importance of the liquidity shocks dominates, and hence banks decide to hold positive precautionary. But as $\zeta(2)$ is sufficiently large, the benefit of holding precautionary is less. This response pattern of the core banks in the crisis regime is consistent with our analysis discussed in Section 4. As expected, core banks reduce their adjustment speeds in the crisis regime as $\zeta(2)$ increases due to the lower benefit. Interestingly, peripheral banks hold positive precautionary liquidity and raise it in response to larger $\zeta(2)$ in both regimes. As mentioned earlier, liquidity shocks tend to be followed by liquidity outflows from the peripheral banks to the core banks via the interbank network, and hence it is optimal for the peripheral banks to hold extra precautionary liquidity as the interbank's interconnectedness level is higher. However, the peripheral banks keep the adjustment speed approximately the same in the normal regime and only slightly decrease it in the crisis regime. In summary, the increase in the interconnectedness level in the crisis regime makes the direct benefit of liquidity provision lower (and hence lower γ_1), but its indirect effect through systemic liquidity shocks makes peripheral banks hold more precautionary liquidity.

5.5 Effects of systemic liquidity shocks

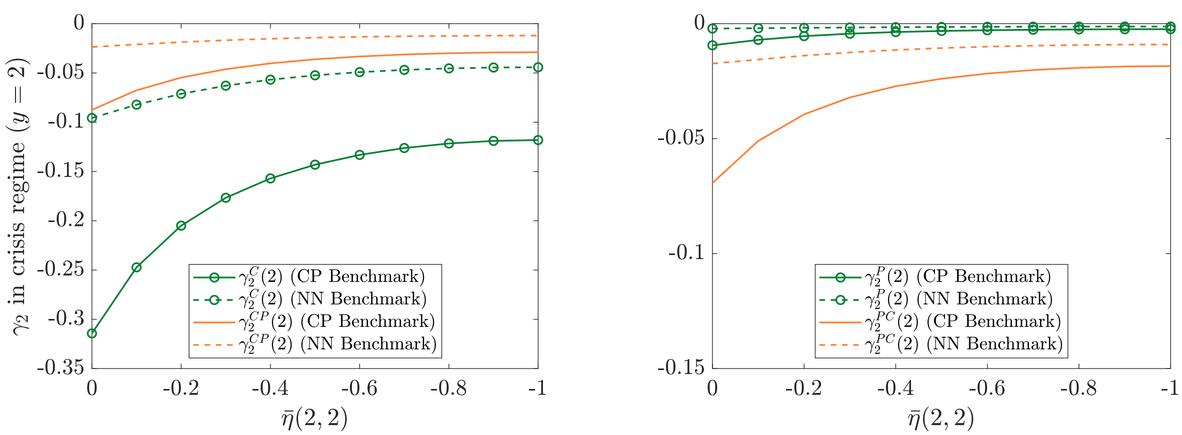
Figure 3 shows the plots of the proportional liquidity provision γ_2 in the normal regime as the size of the normal-to-crisis regime-transition shock ($\bar{\eta}(1,2)$) increases. In this analysis we assume that the relative shock sizes are the same for both core and peripheral banks ($\bar{\eta}_C(1,2) = \bar{\eta}_P(1,2) = \bar{\eta}(1,2)$) so that we can vary the value of the negative shock from 0 to -1. As we can see, both types of banks reduce the proportional liquidity provision in absolute terms in both regimes when the shock size increases. This is consistent with the result in Section 4 where banks accept the consequent of systemic shocks, and respond less aggressively as the shock's effect is larger.

Figure 4 shows the effects of the size of the within-crisis-regime liquidity shock ($\bar{\eta}(2,2)$) on the proportional liquidity provision in the crisis regime, assuming the same relative shock size for both core and peripheral banks ($\bar{\eta}_C(2,2) = \bar{\eta}_P(2,2) = \bar{\eta}(2,2)$). Similar to the case of increasing the regime-transition shock's size, all banks for both benchmark models reduce their proportional



(a) Core bank (b) Peripheral bank

Figure 3. Values of proprotional liquidity provision for the interbank relationship in the normal regime of core banks ($\gamma_2^C, \gamma_2^{CP}$) (left) and of peripheral banks ($\gamma_2^P, \gamma_2^{PC}$) (right) associated with the core-periphery (CP) benchmark (solid line) and the no-network (NN) benchmark (dashed line) models as a function of the size of the normal-to-crisis regime-transion liquidity shock ($\bar{\eta}(1,2)$) assuming the same relative shock size for both core and peripheral banks ($\bar{\eta}_C(1,2) = \bar{\eta}_P(1,2) = \bar{\eta}(1,2)$).



(a) Core bank (b) Peripheral bank

Figure 4. Values of proprotional liquidity provision for the interbank relationship in the crisis regime of core banks ($\gamma_2^C, \gamma_2^{CP}$) (left) and of peripheral banks ($\gamma_2^P, \gamma_2^{PC}$) (right) associated with the core-periphery (CP) benchmark (solid line) and the no-network (NN) benchmark (dashed line) models as a function of the size of the within-crisis-regime liquidity shock ($\bar{\eta}(2,2)$) assuming the same relative shock size for both core and peripheral banks ($\bar{\eta}_C(2,2) = \bar{\eta}_P(2,2) = \bar{\eta}(2,2)$).

liquidity provision for the interbank relationship, but the responses are more sensitive to the shock size, as the arrival rate of this type of shock is larger ($\lambda(2,2) > \lambda(1,2)$).

The effects of the size of the normal-to-crisis regime-transition shock $\bar{\eta}(1,2)$ on proportional precautionary liquidity (upper panel) and non-incentive component of the adjustment speeds (lower panel) are shown in Figure 5. The banks' response pattern for both precautionary liquidity

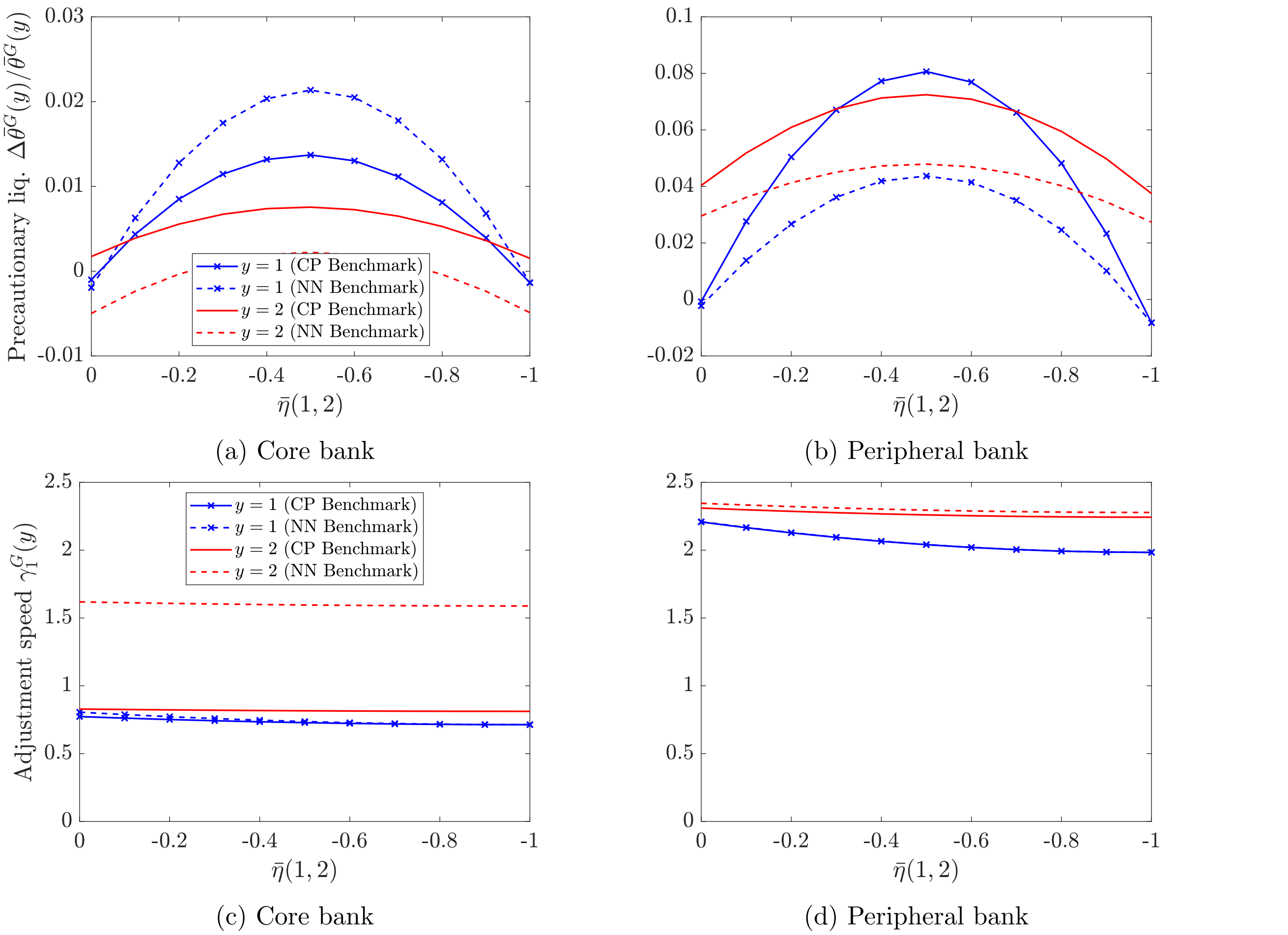
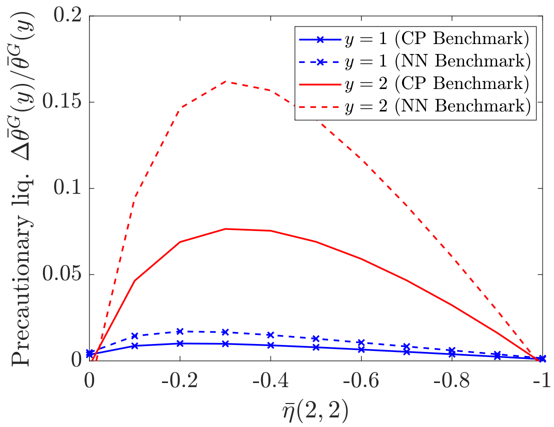
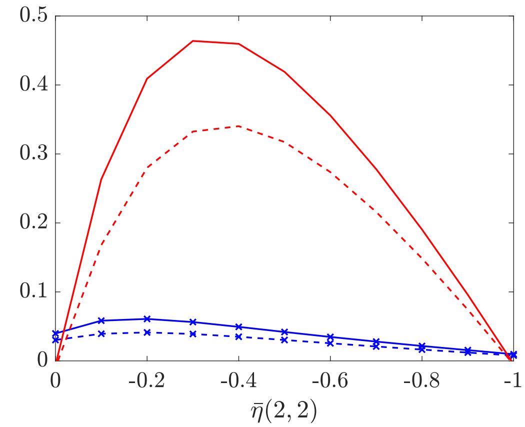


Figure 5. Values of proportional precautionary liquidity ($\Delta\bar{\theta}/\bar{\theta}$) upper) and values of adjustment speed γ_1 (lower) for core (left) and peripheral (right) banks associated with the core-periphery (CP) benchmark (solid line) and the no-network (NN) benchmarkk (dashed line) models as a function of the size of the normal-to-crisis regime-transition liquidity shock ($\bar{\eta}(1, 2)$) assuming the same relative shock size for both core and peripheral banks ($\bar{\eta}_C(1, 2) = \bar{\eta}_P(1, 2) = \bar{\eta}(1, 2)$). The values are for normal (lines with crosses) and crisis (lines without crosses) regimes.

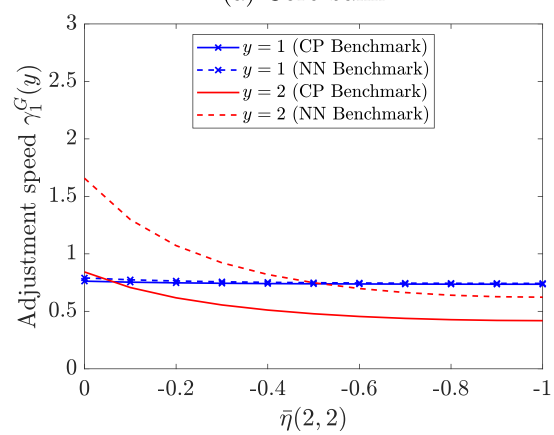
and adjustment speeds are consistent with the analysis in Section 4. That is, as the shock size is larger, banks partly accept the consequence and lower the adjustment speed, but at the same time hold more precautionary liquidity for relatively less negative shock size to reduce its effect (from 0 to around -50% in this case). As the shock size becomes more negative, extra precautionary liquidity would not help as it will be mostly gone when a shock comes. Consistent with earlier observations, core banks hold more precautionary liquidity during the normal regime when the interbank network collapses in the crisis regime (NN benchmark) than when it does not (CP benchmark) because the size of the shock is larger for the core banks than for the peripheral banks, and the interbank channel for liquidity inflows for the core banks breakdowns. The opposite is true for the peripheral banks. The precautionary liquidity in the crisis regime are less sensitive to the shock size $\bar{\eta}(1,2)$, but still quite significant. In the crisis regime, both types of banks hold less precautionary liquidity in the no-network model than in the core-peripheral model because the volatility effect becomes more important as discussed in the benchmark models.



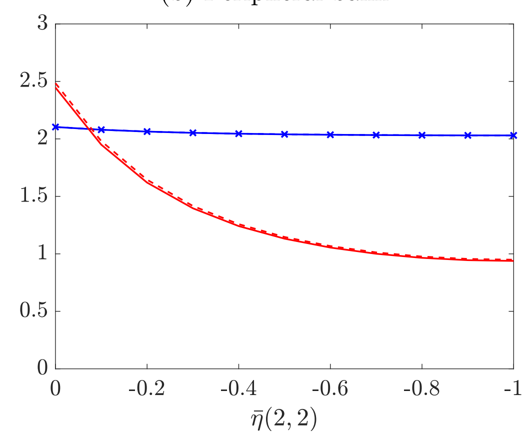
(a) Core bank



(b) Peripheral bank



(c) Core bank



(d) Peripheral bank

Figure 6. Values of proportional precautionary liquidity ($\Delta\bar{\theta}/\bar{\theta}$) (upper) and values of adjustment speed γ_1 (lower) for core (left) and peripheral (right) banks associated with the core-periphery (CP) benchmark (solid line) and the no-network (NN) benchmark (dashed line) models as a function of the size of the within-crisis-regime liquidity shock ($\bar{\eta}(2,2)$) assuming the same relative shock size for both core and peripheral banks ($\bar{\eta}_C(2,2) = \bar{\eta}_P(2,2) = \bar{\eta}(2,2)$). The values are for normal (lines with crosses) and crisis (lines without crosses) regimes.

The effects of the size of the systemic liquidity shock within the crisis regime $\bar{\eta}(2,2)$ are similar to the effects of the normal-to-crisis regime-transition shock $\bar{\eta}(1,2)$ in terms of direction. Figure 6 shows the effects of $\bar{\eta}(2,2)$ on the proportional precautionary liquidity (upper panel) and non-incentive component of the adjustment speeds (lower panel), assuming the same relative shock size for both core and peripheral banks ($\bar{\eta}_C(2,2) = \bar{\eta}_P(2,2) = \bar{\eta}(2,2)$). As we can see from the upper panel, banks hold more precautionary liquidity as the shock size becomes more negative from 0 to around -30% and as the shock size is getting more negative, banks hold less precautionary liquidity. This -30% threshold is less than the threshold in the case of $\bar{\eta}(1,2)$ at -50% in absolute term as the arrival associated with $\bar{\eta}(2,2)$ is more. As shocks happen in the crisis regime, the precautionary liquidity levels for the crisis regime seem to be larger and more sensitive to the shock size. In this analysis, we find that core banks hold more precautionary liquidity in both regimes when the interbank market collapses during the crisis (NN benchmark) than when it does not (CP benchmark), while peripheral banks do the opposite. This again comes from the fact that the absolute impact of the shock on the core banks is larger, and they tend to receive liquidity inflows through the interbank network from peripheral banks. The missing channel in the no-network benchmark model makes the core banks hold more precautionary liquidity. As expected, banks reduce the adjustment speeds as the shock size becomes more negative. The effects are more in the crisis regime, and there are almost no effects on the speeds in the normal regime. Overall, banks have higher adjustment speeds in the no-network model than in the core-periphery model, and the effects for the core banks are larger than those for the peripheral banks as the total level of the interconnectedness of the core banks is more in the normal regime.

5.6 Effects of flight-to-quality flows

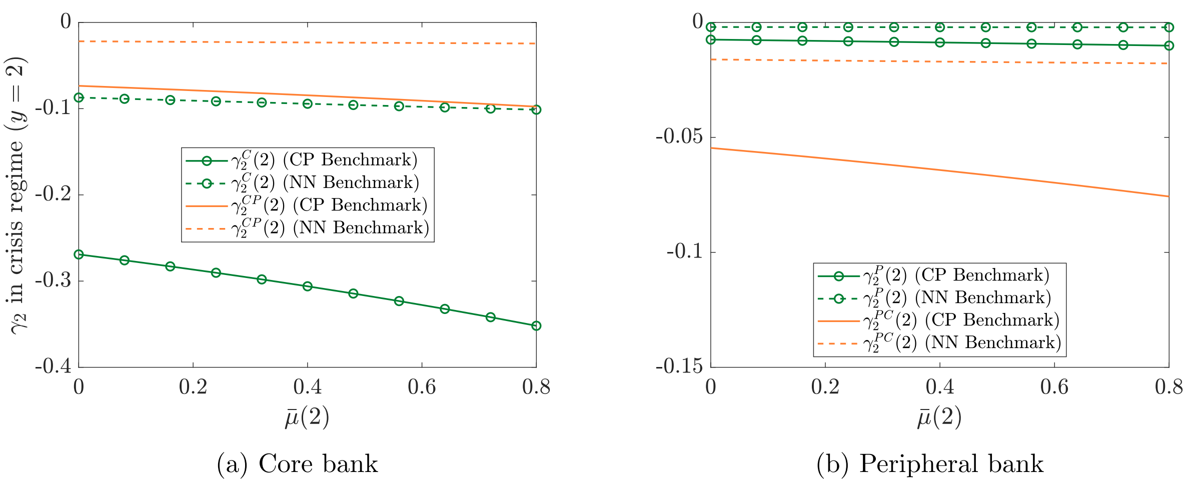
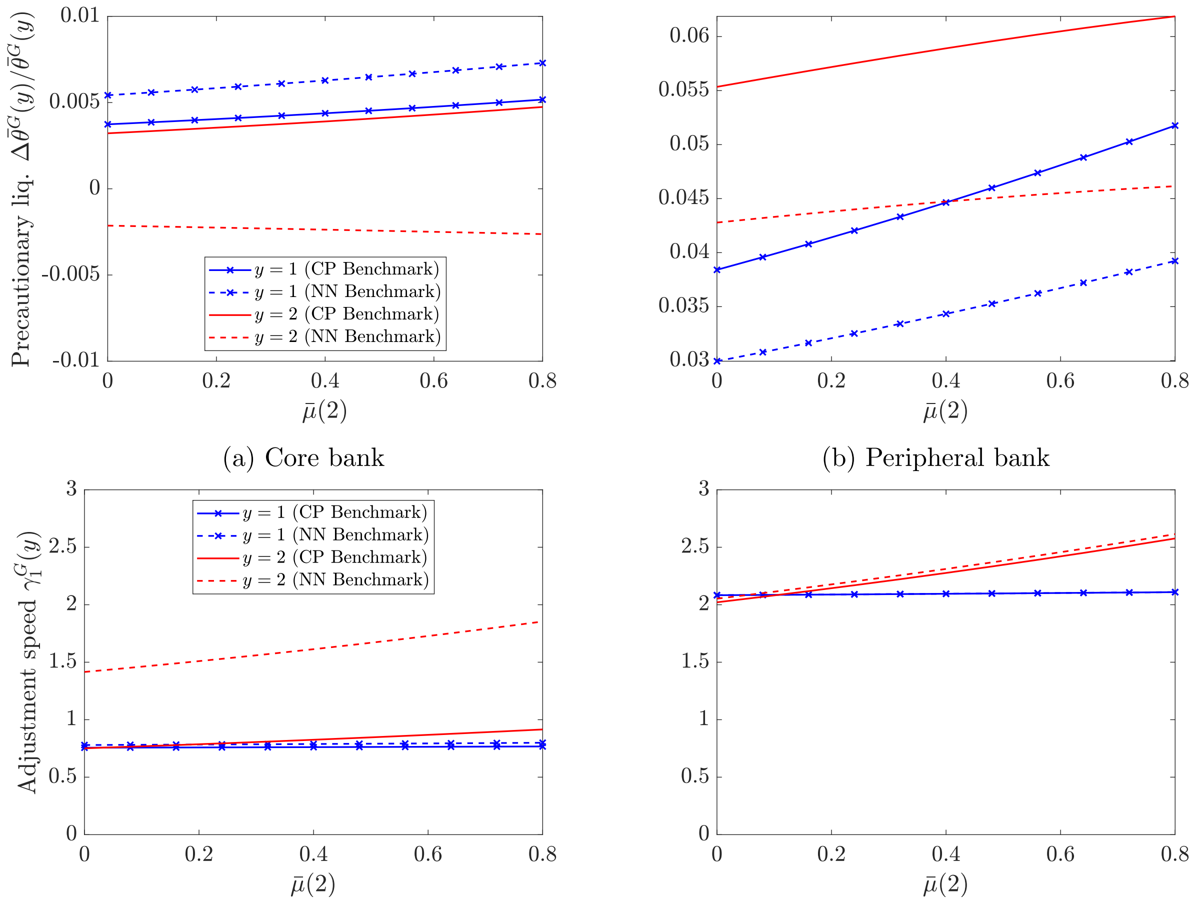


Figure 7. Values of proprotional liquidity provision for the interbank relationship in the crisis regime of core banks ($\gamma_2^C, \gamma_2^{CP}$) (left) and of peripheral banks ($\gamma_2^P, \gamma_2^{PC}$) (right) associated with the core-periphery (CP) benchmark (solid line) and the no-network (NN) benchmark (dashed line) models as a function of the effect of flight-to-quality flows ($\bar{\mu}(2)$) assuming the same effect for both core and peripheral banks ($\bar{\mu}_C(2) = \bar{\mu}_P(2) = \bar{\mu}(2)$).

We study the flight-to-quality effects during crisis, assuming the same flight-to-quality parameter value for both types of banks: $\bar{\mu}_C(2) = \bar{\mu}_P(2) = \bar{\mu}(2)$. We vary its value from 0 to twice the baseline value. Figure 7 shows the effects of $\bar{\mu}(2)$ on the proportional liquidity provision for the interbank relationship (γ_2). As we can see, banks hold more proportional liquidity provision as the effect of the flight-to-quality flows gets stronger, consistent with the results in Section 4. However, the effects under the no-network benchmark model are quite small as the needs for liquidity provision is less when the interbank network collapses in the crisis regime.



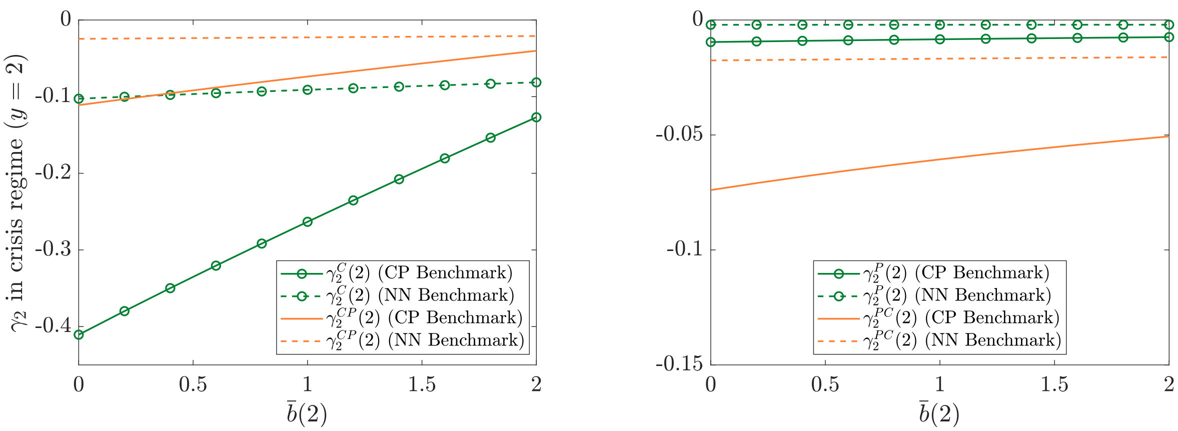
(a) Core bank (b) Peripheral bank

(c) Core bank (d) Peripheral bank

Figure 8. Values of proportional precautionary liquidity ($\Delta\bar{\theta}/\bar{\theta}$) (upper) and values of adjustment speed γ_1 (lower) for core (left) and peripheral (right) banks associated with the core-periphery (CP) benchmark (solid line) and the no-network (NN) benchmark (dashed line) models as a function of the effect of flight-to-quality flows ($\bar{\mu}(2)$) assuming the same effect for both core and peripheral banks ($\bar{\mu}_C(2) = \bar{\mu}_P(2) = \bar{\mu}(2)$). The values are for normal (lines with crosses) and crisis (lines without crosses) regimes.

Figure 8 presents the effects of flight-to-quality flows during the crisis regime, $\bar{\mu}(2)$, on the proportional precautionary liquidity (upper panel) and the non-incentive component of the adjustment speeds (lower panel). The effects on the precautionary liquidity are consistent with the observations made in Section 4. That is, when the liquidity shock has a sufficiently larger effect than the volatility, banks hold positive precautionary and tend to increase it as the degree of the flight-to-quality effect increases. On the other hand, if the volatility's effect is sufficiently larger than the shock's effect, banks hold more negative precautionary liquidity as the flight-to-quality effect is stronger. Banks also increase the non-incentive component of the adjustment speed γ_1 as the the flight-to-quality effect is larger, but the effects in the normal regime are small.

5.7 Effects of reward and penalty parameters

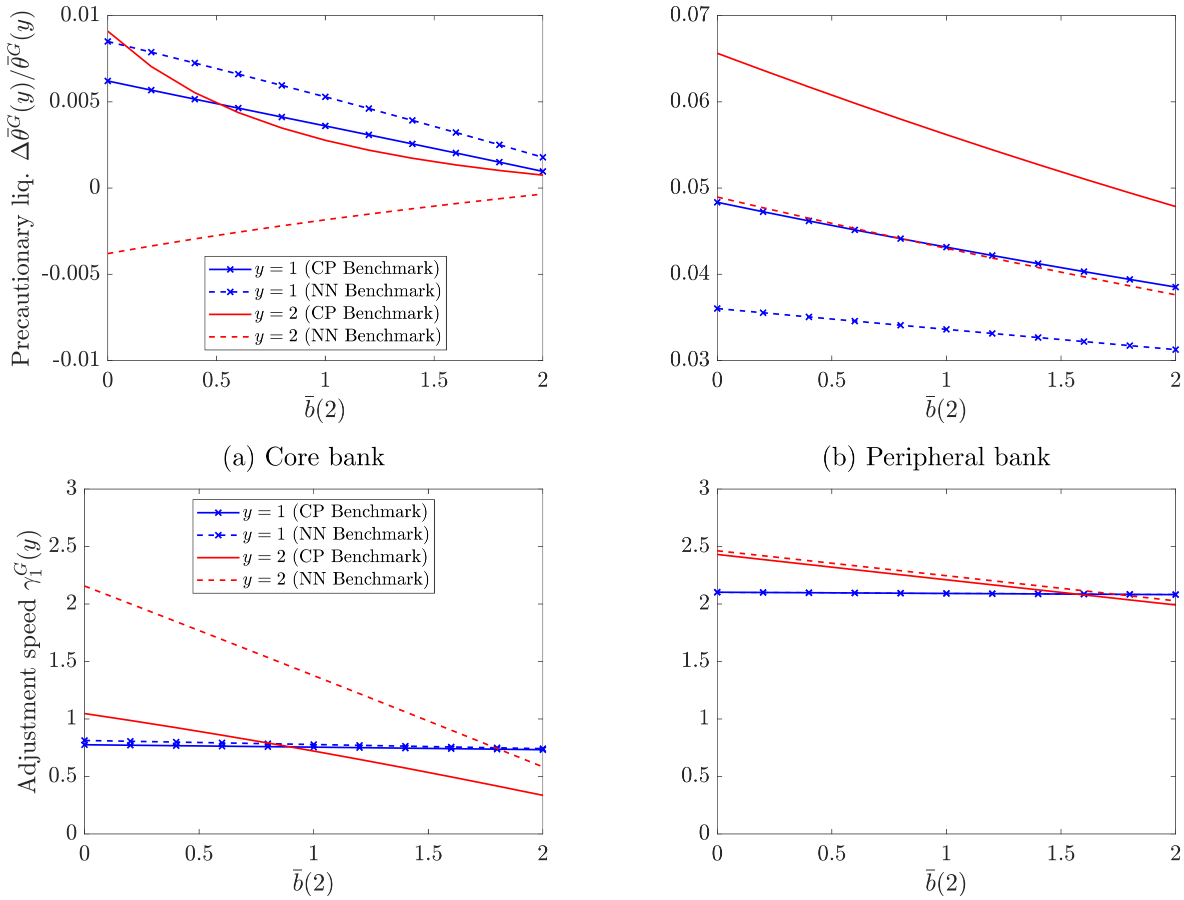


(a) Core bank

(b) Peripheral bank

Figure 9. Values of proprotional liquidity provision for the interbank relationship in the crisis regime of core banks ($\gamma_2^C, \gamma_2^{CP}$) (left) and of peripheral banks ($\gamma_2^P, \gamma_2^{PC}$) (right) associated with the core-periphery (CP) benchmark (solid line) and the no-network (NN) benchmark (dashed line) models as a function of the reward parameter of core banks in the crisis regime $\bar{b}(2)$, assuming the ratio between the parameter values for core and peripheral banks remains the same as in the the baseline value ($\bar{b}_C(2) = \bar{b}(2)$ and $\bar{b}_P(2) = \bar{b}(2) \times 2/7$).

The incentive or reward parameter during the crisis regime as measured by $\bar{b}(2)$ could represents the facilities rate provided by regulators for lending and borrowing liquidity outside the interbank market. In this analysis, we assume that the ratio between the parameter values for core and peripheral banks remains the same as in the the baseline value. That is, we set $\bar{b}_C(2) = \bar{b}(2)$ and $\bar{b}_P(2) = \bar{b}(2) \times 2/7$ and vary the value of $\bar{b}(2)$ from 0 to 2 so that the assumption that $\bar{b}_G(2) < \sqrt{\bar{q}_G(2)}$, $G \in \{C, P\}$, is still satisfied. Under this setting, lending and borrowing liquidity of core banks are relatively better facilitated from the outside interbank market as they have lower risk profiles and higher capacity. Figure 9 shows the effects of $\bar{b}(2)$ on the proportional liquidity provision. As we can see, the results are consistent with the analysis in Section 4. A better facilities rate during the crisis leads to the reduction in the magnitude of the liquidity provision for interbank lending (γ_2). In other words, it implies that banks can quickly offset any liquidity need or surplus due to the interbank relationship through the lending and borrowing facilities at a low price, so high provision for the interbank relationship is not necessary. As expected, the banks' responses are less sensitive to the changes in the reward parameter when the interbank network collapses during the crisis regime.



(a) Core bank

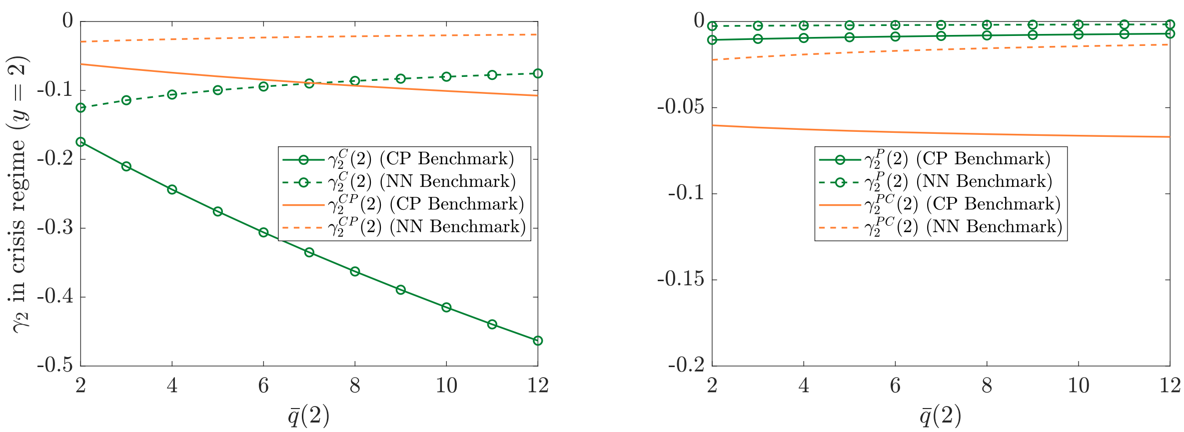
(b) Peripheral bank

(c) Core bank

(d) Peripheral bank

Figure 10. Values of proportional precautionary liquidity ($\Delta\bar{\theta}/\bar{\theta}$) (upper) and values of adjustment speed γ_1 (lower) for core (left) and peripheral (right) banks associated with the core-periphery (CP) benchmark (solid line) and the no-network (NN) benchmark (dashed line) models as a function of the reward parameter of core banks in the crisis regime $\bar{b}(2)$, assuming the ratio between the parameter values for core and peripheral banks remains the same as in the the baseline value ($\bar{b}_C(2) = \bar{b}(2)$ and $\bar{b}_P(2) = \bar{b}(2) \times 2/7$). The values are for normal (lines with crosses) and crisis (lines without crosses) regimes.

Figure 10 shows the effects of the reward parameter in the crisis regime on the proportional precautionary (upper panel) and the non-incentive component of the adjustment speeds (lower panel). As we can see, the proportional precautionary liquidity and the adjustment speeds are reduced in the normal regime when the reward parameter in the crisis regime increases as banks foresee more liquidity facilities during the crisis. This is a moral hazard problem in which banks prepare less for the future shocks if regulators are going to facilitate liquidity needs during crises. For the crisis regime, banks' behavior is the same as in the case of the complete network discussed in Section 4. That is, when the reward parameter increases, banks reduce the non-incentive component of the adjustment speeds and precautionary liquidity in absolute terms, but they increase the total adjustment speeds $\bar{b}_G(2) + \gamma_1^G(2), G \in \{C, P\}$. In other words, banks tend to move their liquidity levels closer to their original targets with higher adjustment speeds to enjoy the high rewards.

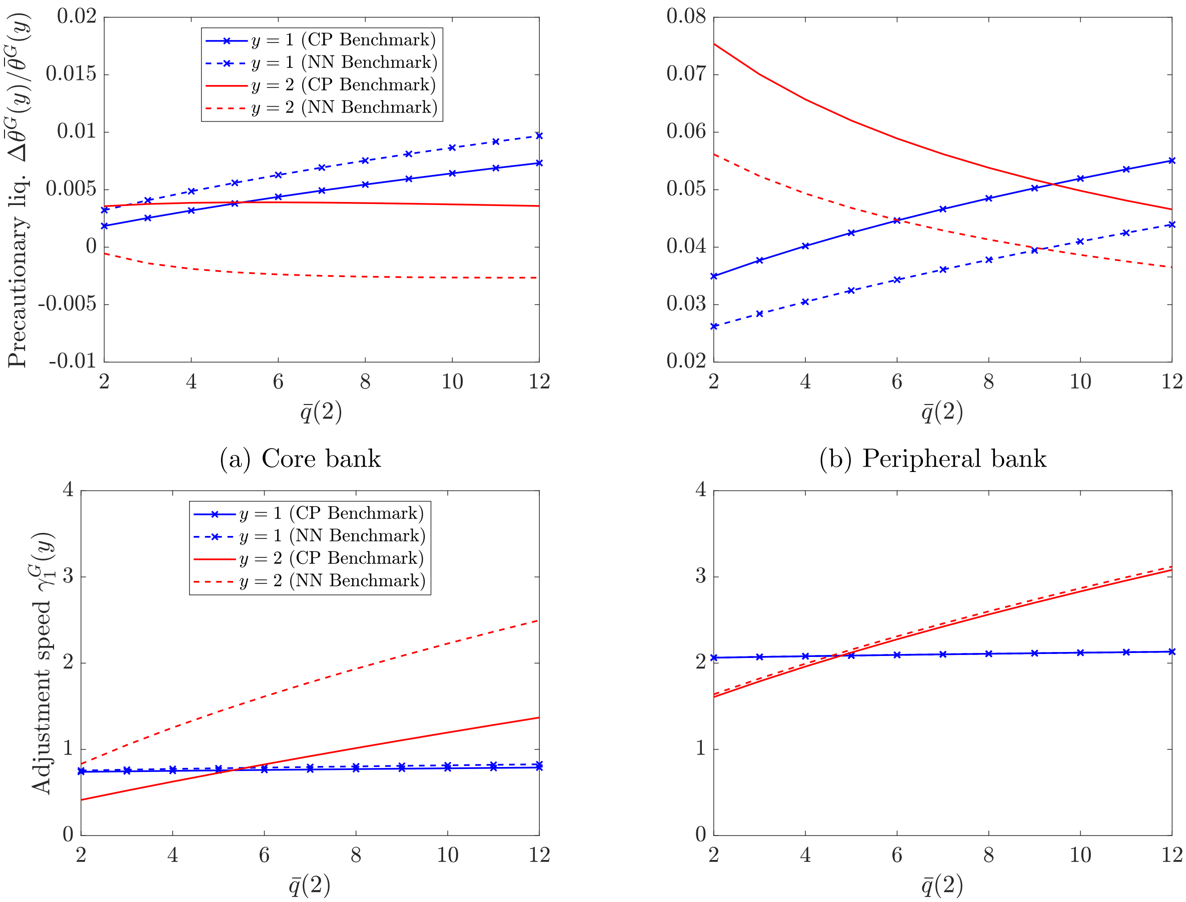


(a) Core bank

(b) Peripheral bank

Figure 11. Values of proprotional liquidity provision for the interbank relationship in the crisis regime of core banks ($\gamma_2^C, \gamma_2^{CP}$) (left) and of peripheral banks ($\gamma_2^P, \gamma_2^{PC}$) (right) associated with the core-periphery (CP) benchmark (solid line) and the no-network (NN) benchmark (dashed line) models as a function of the penalty parameter in the crisis regime ($\bar{q}(2)$) assuming the same parameter for both core and peripheral banks ($\bar{q}_C(2) = \bar{q}_P(2) = \bar{q}(2)$).

Now we consider the penalty parameter in the crisis regime as measured by $\bar{q}(2)$. With the high value of $\bar{q}(2)$, the deviation of liquidity from the target becomes more important, which implies that a higher degree of banks' prudence is required. For instance, a bank with too low liquidity level would suffer a large funding cost from default risk whereas a bank with too high liquidity level would lose opportunities to generate more profit from providing loans. During the crisis, the liquidity level tends to stay below the target due to the systemic liquidity shocks, so $\bar{q}(2)$ could implicitly represent the regulatory costs which penalize banks that likely encountered liquidity problems. We set $\bar{q}_C(2) = \bar{q}_P(2) = \bar{q}(2)$ and vary the value of $\bar{q}(2)$ from 2 to 12 to keep the condition $\bar{q}_G(2) > \bar{b}_G(2)^2$ satisfied for both types of banks $G \in \{C, P\}$. Figure 11 shows the effects of the penalty parameter in the crisis regime $\bar{q}(2)$ on the proportional liquidity provision for the interbank relationship (γ_2). For peripheral banks, the value of γ_2 is quite insensitive to changes in $\bar{q}(2)$ as the degree of interconnectedness is relatively small. However, under the core-periphery benchmark model, core banks implement more prudential control policy; they try to prepare liquidity for interbank lending in order to moderate the peers' influence, especially between core banks themselves so that they can control their liquidity more effectively to reduce the penalty cost. That preparation is reduced if there is no network within the crisis regime. In fact, under the no-network model, core banks reduce their liquidity provision for the other banks in the crisis regime as the penalty parameter increases. Because there is no interbank activities in the crisis regime under the no-network model, the liquidity provision in this regime is just to ensure that when the regime switches back to the normal regime, the liquidity gaps between banks will not be too large. But as the penalty parameter in the crisis regime is larger, banks need to focus more on their current liquidity levels, and hence choose to reduce the interbank liquidity provision to gain more control.



(a) Core bank (b) Peripheral bank

(c) Core bank (d) Peripheral bank

Figure 12. Values of proportional precautionary liquidity ($\Delta\bar{\theta}/\bar{\theta}$) (upper) and values of adjustment speed γ_1 (lower) for core (left) and peripheral (right) banks associated with the core-periphery (CP) benchmark (solid line) and the no-network (NN) benchmark (dashed line) models as a function of the penalty parameter in the crisis regime ($\bar{q}(2)$) assuming the same parameter for both core and peripheral banks ($\bar{q}_C(2) = \bar{q}_P(2) = \bar{q}(2)$). The values are for normal (lines with crosses) and crisis (lines without crosses) regimes.

Figure 12 shows the effects of $\bar{q}(2)$ on the precautionary liquidity (upper panel) and the non-incentive component of the adjustment speeds (lower panel). As we can see, the effects of the penalty parameter $\bar{q}(2)$ are opposite to the incentive parameter $\bar{b}(2)$ due to their different roles on the cost function. Intuitively, a high penalty cost during the crisis regime results in banks' prudential liquidity management, and this leads to a higher target for the precautionary liquidity in the normal period as a buffer of shocks. In the crisis regime, core banks hold approximately the same proportional precautionary liquidity for all values of $\bar{q}(2)$, but increase the adjustment speeds to ensure that their liquidity levels stay close to the targets. Peripheral banks deal with the adjustment speed similarly, but with reduced proportional precautionary liquidity as they know that the liquidity risk arising from their peers (core banks) via the interbank market is mitigated by the highly prudential control policy of the core banks.

6 Systemic liquidity risk

In this section, we study the effects of risk parameters, including the overall interbank network exposure, systemic liquidity shocks, and flight-to-quality flows, and the effects of the reward and penalty parameters on the systemic liquidity risk in the banking system using simulation.

6.1 Systemic liquidity risk indicator

To measure the systemic risk, Borovykh et al. (2018) propose an indicator of systemic risk defined by

$$R(T) = \frac{1}{M} \sum_{i=1}^{M} 1_{\{\min_{0 \le t \le T} X_i(t) < D_i\}} \tag{84}$$

where D_i is the minimum threshold of the liquidity requirement of bank i. If the liquidity of bank i falls below this threshold at any point before or at time T, the bank is considered as a defaulted bank. That is, this indicator measures the default rate in the banking system over a given horizon T. The concept of default rate for systemic risk measurement is also applied in Capponi et al. (2020). However, each of defaulted banks, in fact, contributes its losses to the financial and economic systems differently. In our study, we also use the *weighted default rate* given by

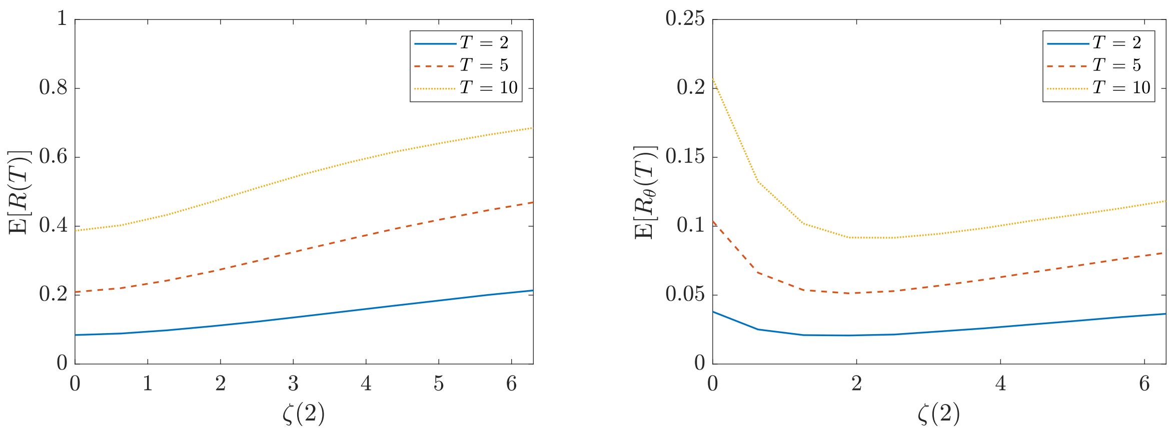
$$R_\theta(T) = \sum_{i=1}^{M} \frac{\theta_i(Y(t)) 1_{\{\min_{0 \le t \le T} X_i(t) < D_i\}}}{\sum_{j=1}^{M} \theta_j(Y(t))}. \tag{85}$$

For this indicator, bank i with liquidity target θ_i contributes its losses to the indicator by the proportion $\theta_i / \sum_{j=1}^{M} \theta_j$ if it is defaulted. Hence, a default of a large bank with large liquidity target θ_i has a high impact on the banking system. In our study, these large banks refer to our core banks in the core-periphery network structure, which play a significant role in the interbank network.

We use the parameters given in Table 1 as our baseline, and consider both of the benchmark models. The first benchmark is the core-peripheral model in which the network structure in both normal and crisis regimes is core-periphery. The second benchmark is the no-network model in which the network structure in the normal regime is core-periphery, but the interbank network collapses during the crisis regime. We assume that banks control their liquidity according to the optimal control policy. We set the initial state of economy as the normal regime ($Y(0) = 1$). The initial liquidity levels of all core banks are assumed to be at their targets $\bar{\theta}_C$. For peripheral banks, we assume that half of them have initial liquidity levels at $\bar{\theta}_P + \bar{\theta}_P/6$, and the initial liquidity levels of the other half are at $\bar{\theta}_P - \bar{\theta}_P/6$. That is, core banks act like intermediary bank in the banking system, while half of the peripheral banks are interbank lenders, and the other half are interbank borrowers. We assume that the minimum threshold of the liquidity requirement is half of the target. Specifically, the threshold of core banks is $D_i = \bar{\theta}_C/2$ for $i \in C$, and that of peripheral banks is $D_i = \bar{\theta}_P/2$ for $i \in P$. We report the average values of the default rate $R(T)$ and weighted default rate $R_\theta(T)$ for $T = 2, 5$ and 10 years to study the impacts on the short, medium and long terms.

6.2 Effects of interconnectedness level

Figure 13 reports the level of systemic risk based on the default rate $R(T)$ (left) and weighted default rate $R_\theta(T)$ (right) as a function of the overall interbank network exposure in the crisis regime, $\zeta(2)$, which contributes to the interconnectedness level of the interbank network, for 2 years (solid line), 5 years (dashed line) and 10 years (dotted line). We vary the value of $\zeta(2)$ from 0 to 3 times of the level in the normal regime. As we can see from the left panel, increases in the exposure always lead to a higher number of defaulted banks for all horizons. This result could come from the fact that shocks are systemic, so the benefit of the interbank network on absorbing shocks for each other is less, but the role of the interbank network as channels for transmitting shocks still exists. In fact, further analysis shows that as the overall network exposure increases, the number of defaulted peripheral banks increases while the number of defaulted core banks decreases. As mentioned earlier, shocks reduce the liquidity of core banks more in absolute terms, and hence peripheral banks are likely to be required to provide liquidity to core banks via the interbank relationship after shocks occur, even though the peripheral banks lost their liquidity by a larger percentage with respect to their liquidity levels. In other words, the losses from core banks are transmitted to peripheral banks that face much worse liquidity shortage. The results for the weighted default rate from the right panel of Figure 13 confirm this. When the overall interbank exposure is low, the weighted default rate is decreasing in the exposure due to the less number of defaulted core banks, and at the cost of defaulted peripheral banks. Interestingly, as the level of the overall network exposure increases further, the weighted default rate starts to increase. This is possibly because the benefit that core banks receive has reached its limit; the exposure level is already large enough to help save all (or most) of the core banks. However, as long as the core banks still remain in the system, a larger rate of shock transmission continues to push the number of defaulted peripheral banks up. As we can see, the level of the overall interbank exposure that yields the lowest weighted default rate depends on the horizon. As the horizon is longer, the *optimal* exposure level seems to be higher.



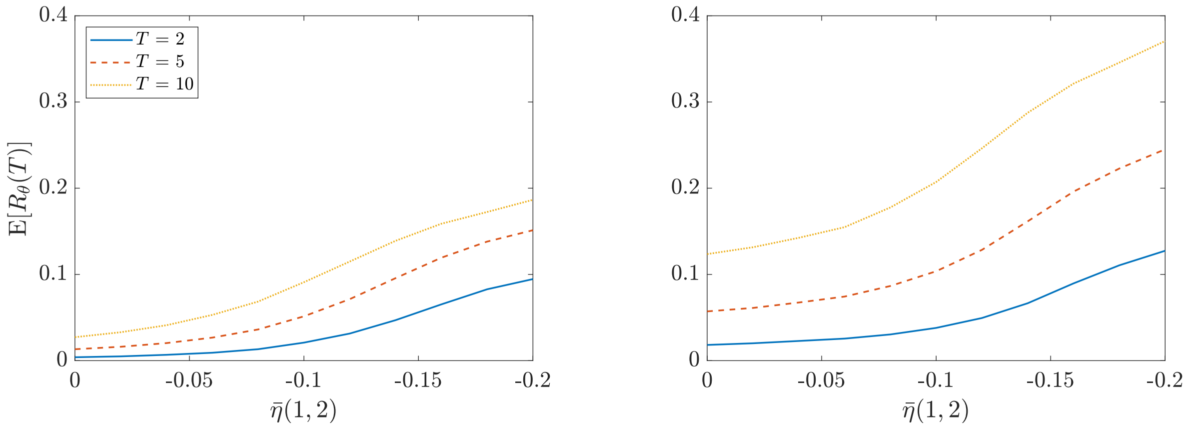
(a) Default rate (b) Weighted default rate

Figure 13. Expected values of default rate $R(T)$ (left) and weighted default rate $R_\theta(T)$ (right) associated with the core-periphery benchmark model as a function of $\zeta(2)$. The indicator is shown for $T = 2$, 5 and 10 years.

6.3 Effects of systemic liquidity shocks

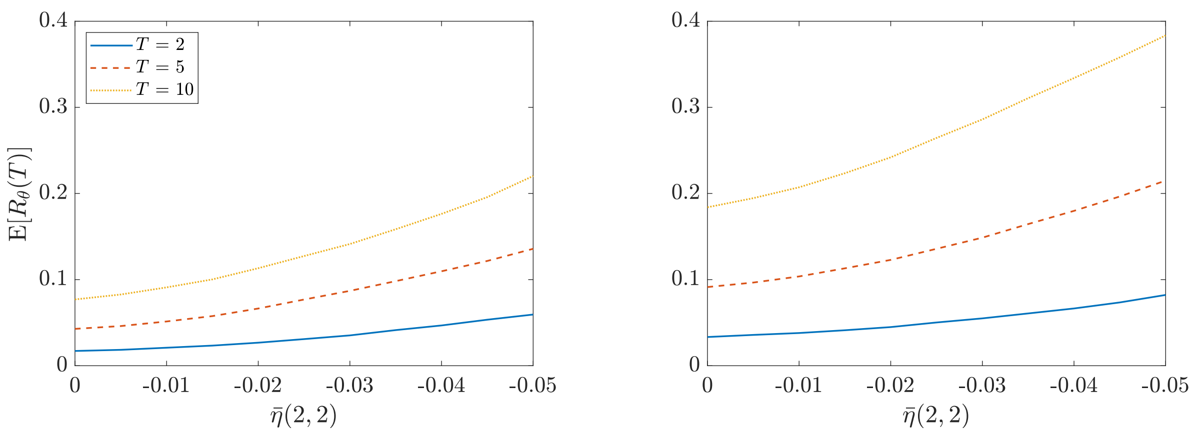
The value of $\eta(1,2)$ measures the size of the systemic liquidity shock initiating a crisis event. We vary the shock size of core banks, $\bar{\eta}_C(1,2)$, up to -20% and assume that the peripheral banks have 3 times larger shock size to keep the ratio of the shock sizes the same as that in the benchmark models. That is, $\bar{\eta}_P(1,2) = 3 \times \bar{\eta}_C(1,2)$. Figure 14 shows the results for the core-periphery (left) and no-network (right) models. A larger shock triggers more systemic risk, but the relationship is not linear. The systemic risk indicator seems to be less sensitive for small shock sizes. The weighted default rate accelerates when the shock size of the core banks is around 8% - 15%, and it is slowing down when the shock size is larger than 15% in magnitude. In this situation, the default rate is mainly contributed from peripheral banks, since they not only suffer from their individual shocks but also from core banks' liquidity needs as mentioned above. The shock size of 15% in core banks means 45% in peripheral banks, so it causes a high default rate in the peripheral banks. When many of the peripheral banks have already defaulted, the number of the non-defaulted banks reduces. As a consequence, the rate at which new defaults occur is low.

Comparing the two benchmark models, we find that the systemic risk on the no-network model is larger as the benefit for the core banks disappears during the crisis regime.



(a) Core-peripheral benchmark model (b) No-network benchmark model

Figure 14. Expected values of weighted default rate $R_\theta(T)$ associated with the core-peripheral benchmark (left) and the no-network benchmark (right) models as a function of $\bar{\eta}(1,2)$ where $\bar{\eta}_C(1,2) = \bar{\eta}_P(1,2)/3 = \bar{\eta}(1,2)$. The indicator is shown for $T = 2$, 5 and 10 years.

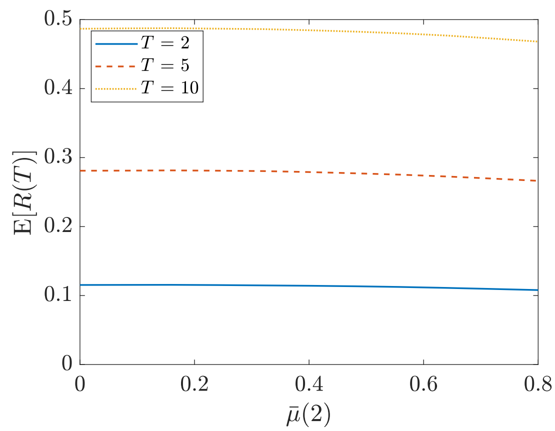


(a) Core-peripheral benchmark model (b) No-network benchmark model

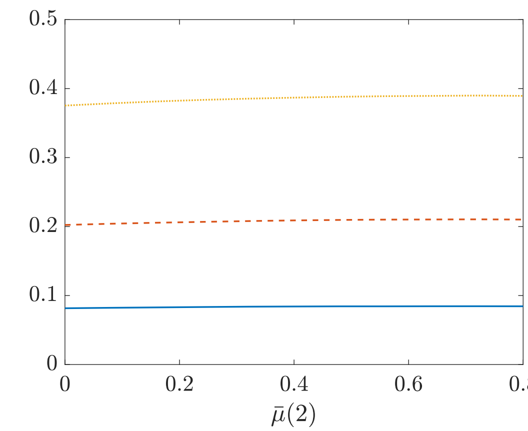
Figure 15. Expected values of weighted default rate $R_\theta(T)$ associated with the core-peripheral benchmark (left) and the no-network benchmark (right) models as a function of $\bar{\eta}(2,2)$ where $\bar{\eta}_C(2,2) = \bar{\eta}_P(2,2)/3 = \bar{\eta}(2,2)$. The indicator is shown for $T = 2$, 5 and 10 years.

We now consider the effects of the size of the within-crisis shocks $\bar{\eta}(2,2)$. Similarly to the case of $\bar{\eta}(1,2)$, we assume that the shock size for the peripheral banks is 3 times as large as the size of the shock for the core banks. Figure 15 shows the results for both benchmark models. As expected, the systemic risk increases as the shock size becomes more negative. Since the within-crisis regime shock has a higher arrival rate compared to the regime-transition shock, the within-crisis regime shock causes a higher weighted default rate given the same shock size. In this study, we find that the within-crisis regime shocks trigger more defaulted core banks than the other type of shocks.

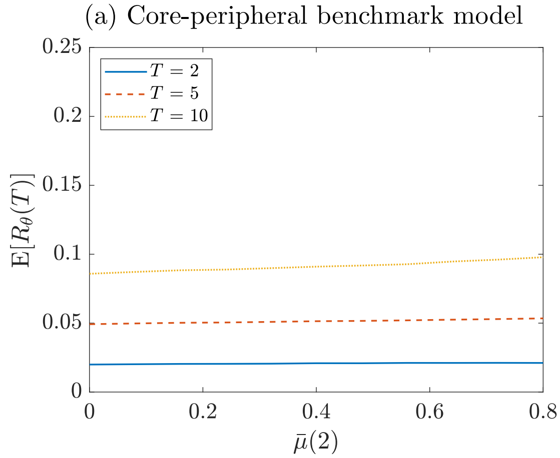
6.4 Effects of flight-to-quality flows



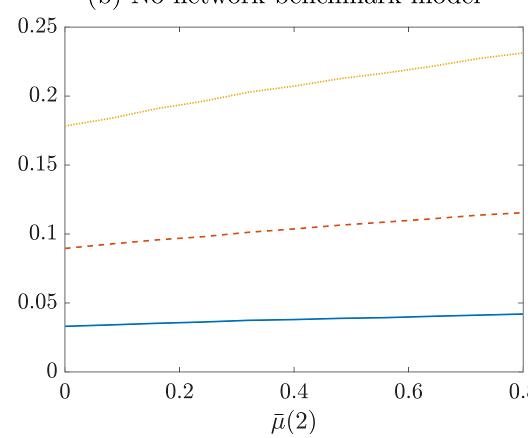
(a) Core-peripheral benchmark model



(b) No-network benchmark model



(c) Core-peripheral benchmark model



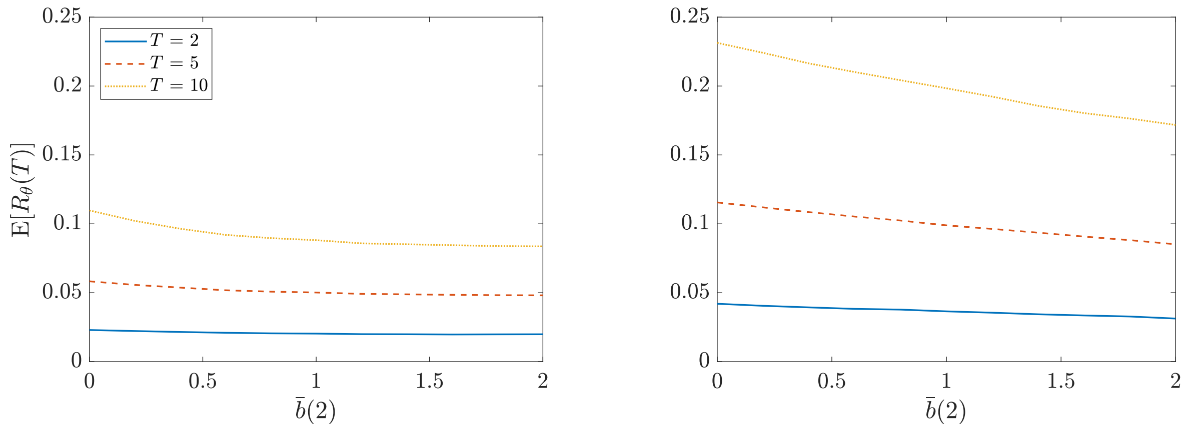
(d) No-network benchmark model

Figure 16. Expected values of defaulte rate $R(T)$ (upper) and weighted default rate $R_\theta(T)$ (lower) associated with the core-peripheral benchmark (left) and the no-network benchmark (right) models as a function of $\bar{\mu}(2)$ where $\bar{\mu}_C(2) = \bar{\mu}_P(2) = \bar{\mu}(2)$. The indicator is shown for $T = 2$, 5 and 10 years.

In our model, the flight-to-quality flows represents the effects of panic-triggered flows of liquidity during crises, the degree of which is measured by $\bar{\mu}(2)$. In this analysis, we assume that all banks have identical degree of flight-to-quality effect ($\bar{\mu}_C(2) = \bar{\mu}_P(2) = \bar{\mu}(2)$), and report the simulation results in Figure 16 for both expected default rate $R(T)$ (upper panel) and expected weighted default rate $R_\theta(T)$. As we can see, the expected default rates do not change much as $\bar{\mu}(2)$ increases. In fact, the default rates drop slightly as the effect of the flight-to-quality flows becomes stronger in the core-periphery benchmark model. From the results in Section 5.6, we find that banks hold positive precautionary liquidity and the level goes up as $\bar{\mu}(2)$ increases under the core-periphery model (see Figure 8). So banks tend to control their liquidity more aggressively given a high level of the flight-to-quality effect, and consequently, the default rate slightly decreases. However, the expected weighted default rate slightly increases as $\bar{\mu}(2)$ increases. The effects on the weighted default rate become more pronounced under the no-network benchmark model. The results in Section 5.6 show that core banks hold negative precautionary liquidity in the crisis regime under the no-network benchmark model. In addition, they increase the adjustment speed γ_1 and reduce the precautionary liquidity further as the degree of the flight-to-quality effect increases. Hence, when the flight-to-quality effect is large, and the interbank market does not function, core banks tend to lose more liquidity as they are not allowed to utilize relatively higher (or less negative) excess liquidity from the peripheral banks. As a result, the weighted default rate goes up at a faster rate than the default rate under the no-network benchmark model.

6.5 Effects of the reward and penalty parameters

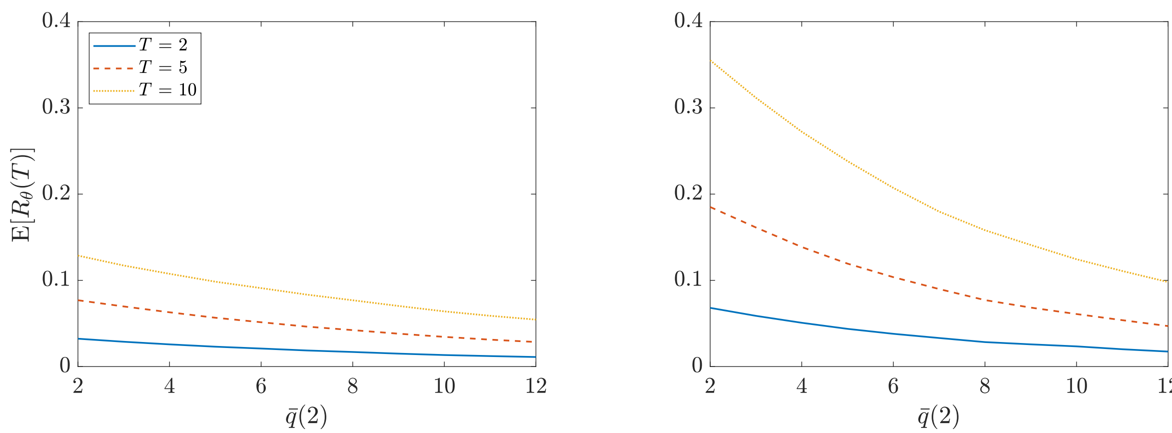
When regulators plan to help banks manage their liquidity during the crisis regime by providing lending facilities with incentive parameter $\bar{b}(2)$, banks adversely respond to this effect creating a moral hazard problem. Figure 17 shows the weighted default rates as a function of the incentive parameter in the crisis regime or $\bar{b}(2)$. As we can see, this policy slightly relieves the systemic risk under the benchmark model, and the benefits are apparent in long term under the no-network benchmark model due to the utilization of regulators' facilities without interbank noises. However, this regulatory policy could be inefficient since the costs of implementing the policy might not be worth for the marginal benefits. Namely, the policy does not encourage banks to prepare for and be conscious of the liquidity risks, and related regulators also need to provide the liquidity instead when the liquidity problems emerge. It is worth noticing that the effects of $\bar{b}(2)$ and $\bar{\mu}(2)$ seems to be reversal as they are transmitted through the model in similar mechanic but different directions. In contrast to providing lending facilities, it should be more valuable to promote banks' own prudential liquidity control policy. As discussed in Section 5.7, increases in $\bar{q}(2)$ lead banks to preserve liquidity buffer in the normal period and try to increase the speeds of adjustment during the crisis regime. It can be seen that the degree of systemic risk is reduced notably as shown in Figure 18.



(a) Core-peripheral benchmark model

(b) No-network benchmark model

Figure 17. Expected values of $R_\theta(T)$ associated with the core-peripheral benchmark (left) and the no-network benchmark (right) models as a function of $\bar{b}(2)$ where $\bar{b}_C(2) = \bar{b}(2)$ and $\bar{b}_P(2) = \frac{2}{7}\bar{b}(2)$. The indicator is shown for $T = 2$, 5 and 10 years.



(a) Core-peripheral benchmark model

(b) No-network benchmark model

Figure 18. Expected values of $R_\theta(T)$ associated with the core-peripheral benchmark (left) and the no-network benchmark (right) models as a function of $\bar{q}(2)$ where $\bar{q}_C(2) = \bar{q}_P(2) = \bar{q}(2)$. The indicator is shown for $T = 2$, 5 and 10 years.

7 Policy Implications and Conclusions

In this study we propose a new interbank network model the allows us to model different types of liquidity risk, including (i) the idiosyncratic shocks from daily deposits and withdrawals, (ii) systemic shocks such as bank runs, (iii) reduction in the interconnectedness in the interbank market during a crisis, and (iv) panic-triggered liquidity outflows following large shocks. We study the effects of the last three types of risk on the optimal liquidity control policy. We find that the optimal control consists of two main components. The first component is the liquidity provision for the interbank relationship. Banks hold liquidity provision according to their current role in the network. Namely, a net lender in the interbank market holds positive provision, while a net borrower holds negative provision. Holding more provision reduces the impacts from the peers and hence allows banks to have more control over their liquidity levels. The second component is the precautionary liquidity. Banks hold positive precautionary liquidity to reserve liquidity for their future uses, including absorbing unexpected losses. There are two key parameters for this precautionary liquidity: the liquidity target and the speed at which banks control their liquidity toward the target.

Different types of risk have different implications on banks' optimal control policies. Banks use liquidity provision to reduce the impact from the interbank relationship, but when the cost of doing so is too high banks decrease the provision. Similarly, banks hold positive precautionary liquidity when they expect higher risk, but when risk is too high, banks lower their precautionary holding. In our analysis of large homogeneous complete network with one regime, we find that banks tend to hold positive precautionary liquidity for systemic shocks, but reduce the liquidity provision and the adjustment speeds. In fact, when the jump size is sufficiently large, banks start to reduce their precautionary as most of the liquidity reserve will be lost once a shock occurs. This creates a challenging problem for regulators. That is, holding positive precautionary liquidity makes the system more resilient to shocks, but when large shocks are expected, banks choose to reduce their reserves.

On the other hand, banks tend to respond to panic-triggered flows or flight-to-quality effects differently. Our analysis shows that banks should increase their liquidity provision and increase the adjustment speed in response to a larger effect of the flight-to-quality flows. However, our numerical examples show that banks tend to reinforce their precautionary liquidity position. As a consequence, when they hold negative precautionary liquidity which is more likely to happen when the interbank market collapses, they tend to reduce their liquidity targets further more as the effect of the flight-to-quality becomes stronger. With an increasing adjustment speed, the banks' liquidity levels drops quickly, causing a larger risk to the banking system. In the case of large homogeneous complete network with one regime, it is optimal for banks to hold negative precautionary liquidity when systemic shocks are very small or very large. So the combination of large shocks and a strong effect of flight-to-quality during crises can trigger this type of events, resulting in larger systemic risk.

When the interbank market collapses during a crisis, losses from one bank cannot be transmitted to the other banks via the interbank market. When losses are systemic, our examples show that the interbank market can transmit losses from unhealthy banks to more unhealthy banks, resulting in a high default rate. The collapse of the interbank market during the crisis lowers the number of defaulted banks. However, from our examples, the total loss is larger when the interbank market does not function during the crisis as larger banks tend to default more. In fact, our examples show that there is an optimal level of interconnectedness that yields the lowest loss in the system, and the optimal level depends on the time horizon. However, the optimal level for the loss does not correspond to the optimal level for the default rate. Determine a way to control the interconnectedness level during crises to result in a good balance between the loss rate and the default rate can be a very challenging task for regulators.

We study two mechanics that can reduce systemic risk. The first mechanic is to provide incentive for banks to increase their liquidity by lowering the cost of lending and borrowing through, for example, regulators' lending facilities. This, however, creates a moral hazard problem as banks do not implement appropriate risk management policy, but plan to use the facilities to adjust their liquidity when needed. Although this can reduce the systemic risk of the banking system, it comes at the cost of the regulators. The second mechanic is through the use of penalty. Regulators may impose penalty for banks that do not hold sufficient liquidity. Our analysis shows that with a larger penalty cost, banks implement more prudential liquidity control policy: they hold positive precautionary liquidity during the normal times and maintain their adjustment speed, and they hold little precautionary liquidity but at a much higher speed to ensure that their liquidity levels do not deviate from their targets. This overall reduces the systemic risk, and, of course, it comes at the cost of the banks. Whether this larger cost for the banks would lead to another type of problems is an interesting issue to investigate but is beyond our current study.

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